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« Equilibrium Lottery Games and Preferences Under Risk »

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Equilibrium Lottery Games and Preferences Under Risk

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Abstract

This article investigates which forms of preferences under risk are consistent with profit-maximizing lotto games. To do so, I depart from the approach employed by Friedman and Savage (1948) by introducing a reference point against which changes in wealth are valued and probability transformation. Both alterations make the model directly comparable to experimental results found in the context of Cumulative Prospect Theory. By exploiting some basic stylized facts about the prize distribution of lotto games, a number of restrictions on preferences are found. The value function must be globally concave over the domain of positive prizes offered by the lottery, and the probability weighting function concave for low probabilities in the gain domain and concave in average for low probabilities in the loss domain. Remarkably, all these properties match the corresponding results found in the experimental literature.

JEL codes : D81, D21
Keywords : Decision-Making under Risk, Lottery Games, Firm Behavior

Résumé :

Cet article s’intéresse à la forme prise par les préférences dans le risque sous l’hypothèse qu’une entreprise vend un jeu de loto de façon à maximiser son profit. Je suis l’approche employée par Friedman et Savage (1948) avec deux exceptions : j’introduis un point de référence pour valoriser les gains et les pertes et je permets une transformation des probabilités. Ces deux enrichissements permettent une comparaison directe avec les préférences trouvées dans le cadre de la Cumulative Prospect Theory. En exploitant un certain nombre de faits stylisés caractérisant les jeux de loto, je montre que la fonction valeur doit être globalement concave dans

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1 Introduction

Lotto games are very popular over the world. They are easy to play and offer a broad range of prizes. In a standard 6/49 game, players have to select six numbers from a field of 49 numbers and match all of them to win the jackpot. Smaller prizes are also available for anyone who matches fewer numbers. In France, annual sales of lotto games (the classic Lotto game and the Euromillion lottery) are larger than one billion euros (Roger, 2009). Lotto games in UK totalled £3.9 billion in 2008\(^1\).

A long-standing question in risk theory is why players are willing to gamble in a market where the lottery operator redistributes around 50% of sales to winners\(^2\). Several behavioral explanations have been proposed to account for the propensity to gamble at unfavorable odds. Friedman and Savage (1948) argue that people are risk-averse when stakes are small or negative but are risk-seeking with respect to prizes that are large enough to elevate their social standing. This inclination is modeled by a reverse S-shaped utility function and amounts to assuming increasing marginal utility for a broad range of wealth. Another line of explanation emphasizes the role of misperception of probabilities. Small probabilities of winning large prizes are overestimated while medium probabilities of loss are more accurately assessed. Probability distortion is allowed by rank dependent expected utility models developed among others by

\(^1\)According to the 2009 Annual report of the operator of the U.K. National Lottery.
\(^2\)See Chen and Chie (2008) for some facts about takeout rates in various countries.
Quiggin (1982) or by the original prospect theory (Kahneman and Tversky, 1979) and the cumulative prospect theory (Tversky and Kahneman, 1992).

By focusing on players’ preferences however, the supply side of the market is generally overlooked. Casual observation shows that games are designed by profit-oriented companies which are keen to make lottery games attractive to players. They regularly experiment new forms of games. For instance, Euromillion, a lotto games endowed with higher jackpots and longer odds has been successfully launched in 2004 in nine European countries. Clotfelter and Cook (1989, 1993) and Walker and Young (2001) provide many illustrations of the way lottery companies shape the payoff distribution in order to enhance their profit.\footnote{Friedman and Savage (1948) made earlier a similar point: "[Lotteries] have been conducted in many countries and for many centuries, so that a great deal of evidence is available about them; there has been extensive experimentation with the terms and conditions that would make them attractive, and much competition in conducting them, so that any regularities they may show would have to be interpreted as reflecting corresponding regularities in human behavior."}

This article builds on this observation by investigating which preference patterns are compatible with the existence of lottery games shaped by a profit-maximizing company. Players are assumed to value gambles according to Cumulative Prospect Theory (Tversky and Kahneman, 1992) except that no a priori assumptions are made regarding the curvature of the three functions that characterize players’ preferences: the value function and the probability weighting functions in the loss domain and in the gain domain. Instead, the sign of the second derivatives of these functions will be endogenously derived by studying their compatibility with the existence of optimal lottery games. We may however notice that such preferences are flexible enough to allow for the presence
of the two above-mentioned explanations of gambling: risk loving through the convexity of the value function and overestimation of small probabilities through the concavity of the probability weighting function in the gain domain.

I begin by outlining three stylized facts about the probability distribution of lotto games: (i) there is a unique level of loss, the ticket price, that occurs with a positive probability mass, (ii) the prize payout distribution of lotto games is best represented by a continuous distribution above a non-negative minimal prize and (iii) the probability density function of money prizes is decreasing over the whole range of gains. I then assume that a monopolistic firm implements a lottery game by optimally choosing the amount of prizes and their probabilities. Once lottery’s optimality conditions are characterized, the problem is reversed by asking which preference patterns are compatible with the existence of optimal games that satisfies the above-mentioned properties. Compelling restrictions on preferences are found. The value function is globally concave over the domain of positive prizes offered by the lottery, and the probability weighting function is concave for low probabilities in the gain domain and concave in average for low probabilities in the loss domain. The concavity of the weighting function in the gain domain for low probabilities ensures that the lottery operator offers a continuous interval of gains. The average concavity of the weighting function in the loss domain for low probabilities is necessary for the operator to set a unique level of loss. Remarkably, these properties are consistent with two central assumptions of Cumulative Prospect Theory: the value function is concave for gains, and the probability weighting functions in the gain domain and in the loss domain are inverse S-shaped.
This article is related to the literature in risk preferences in several ways. Its central assumption is that the lottery operator maximizes its profit when designing the shape of the game. The profit maximization assumption is ubiquitous in all fields of economics but, paradoxically, has not permeated the analysis of lottery products. A few articles exist however, which make lottery games endogenous by assuming that firms maximize their profit. A notable reference is the seminal article by Friedman and Savage (1948) who first suggest that optimal lottery games may reveal interesting information about preferences under risk. In particular, a concave utility function for the richest part of the population is justified in their article by explicitly appealing to the operation of an entrepreneur conducting a lottery and seeking to maximize its profit. While fruitful, their approach suffers from the assumption that risk preferences are solely explained by attitude toward wealth. As a result, they account for very limited patterns of gambling. The present paper essentially adds non-linear probability weighting functions to their approach. This allows to account for a richer set of preferences under risk. Contrary to Friedman and Savage who restrict their analysis to two-payoff lotteries, I show that modelling lotteries endowed with a continuum of gains excludes the possibility of a convex value function in the gain domain whatever the shape of the probability weighting functions.

The model is also close to Quiggin (1991). He considers the optimal structure of a lottery when agents’ preferences are ordered according to a rank dependent utility criterion. Assuming a concave utility function and a reverse S-shaped

\footnote{Their conclusions have also been criticized on other grounds. See Markowitz (1952), Yaari (1965) and Machina (1982).}
probability transformation function, he shows that an optimally designed lottery involves at least two distinct prizes. The class of lotteries he examines is however restricted by three simplifying assumptions: the company breaks even, the number of prizes is not endogenous and all prizes are equally probable. In contrast, an entire section is devoted to the analysis of optimal lottery games in which no prior restrictions on the probability distribution of prizes are made. Maeda (2008) proposes a model of optimal lottery design for public financing by assuming that agents purchase lottery tickets as a way of entertainment. Both articles focus on the shape of optimal lottery games whereas I essentially focus on consequences for risk preferences of the existence of optimal games. The present model only deals with a subclass of lottery games in which actual levels of gains are random due to the parimutuel nature of lotto games. In a companion paper, Direr (2010) investigates preferences under risk for lottery games endowed with guaranteed money prizes and shows that underlying preferences are sensitive to the type of lottery games considered.

By exploiting some basic stylized facts about lotto games and assuming profit maximization, an important goal of the paper is to differentiate the two main factors behind the purchase of lotto tickets, which are the pecuniary motive (modeled through a convex value function) and the probability distortion argument (represented by non-linear weighting functions). This objective is germane to the experimental literature that estimates preference functions for selected groups of respondents (e.g. Gonzalez and Wu, 1999 or Abdellaoui, 2000). The experimental method is however best fit for limited monetary incentives and moderately small probabilities. Results cannot easily be extrapolated to the
purchase of real-world lotteries endowed with large-scale prizes and very long odds. Although the preference elicitation method proposed in this article is different in essence, it is consistent with experimental evidence. Numerous articles in the experimental field have repeatedly found a concave value function in the gain domain (Tversky and Kahneman, 1992; Abdellaoui, 2000; Abdellaoui, Bleichrodt, and Paraschiv, 2004; Gonzalez and Wu, 1999; Wu and Gonzalez, 1996) and an inverse S-shaped probability weighting function both in the gain and in the loss domain (Abdellaoui, 2000; Bleichrodt and Pinto, 2000; Camerer and Ho, 1994; Etchart-Vincent, 2004; Gonzalez and Wu, 1999; Wu and Gonzalez, 1996).

The presentation is organized as follows. Section 2 outlines a few stylized facts about lotto games. Section 3 analyzes a general lottery problem with an arbitrary prize distribution. Section 4 uses the model of Section 3 and derives implications for preferences under risk by exploiting the stylized facts presented in Section 2. The last section provides concluding remarks.

2 The prize distribution of Lotto games

This section provides some stylized facts regarding the prize payout distribution of lotto games that will be useful for the next sections. For concreteness, I will present the prize structure of the French lotto game called loto exploited by the state-owned lottery operator La Française de Jeux. The Loto is a typical 6/49 game: players have to pick six numbers from a field of 49\(^5\). A seventh number (the bonus number) is drawn, which, if found, allows already-winning players

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\(^5\)The rules have been marginally modified in October 2008. I present the rules that were valid before this date.
to enhance their gains. The probability of matching \( n \) winning numbers with or without the bonus number are indicated in Table 1, where \( n+ \) indicates that the bonus number is correctly guessed along with \( n \) winning numbers. They are given for a typical purchase, namely a lottery card with eight grids to fill in, valid for two consecutive drawings the same day (the possibility to win in both drawings is neglected):

<table>
<thead>
<tr>
<th>rank</th>
<th>numbers</th>
<th>probability</th>
<th>average gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>( 1.144 \times 10^{-6} )</td>
<td>1191889</td>
</tr>
<tr>
<td>2</td>
<td>5+</td>
<td>( 6.865 \times 10^{-6} )</td>
<td>15653</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>( 2.883 \times 10^{-4} )</td>
<td>1005</td>
</tr>
<tr>
<td>4</td>
<td>4+</td>
<td>( 6.878 \times 10^{-4} )</td>
<td>44</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>( 1.479 \times 10^{-2} )</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>3+</td>
<td>( 1.957 \times 10^{-2} )</td>
<td>5.2</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>0.2809</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Table 1. Probabilities of winning and average gains in the French Lotto game

Average gains are computed from a dataset of gains that records the money prizes of 466 consecutive drawings run between 01-03-2004 and 03-25-2006\(^6\). While the number of winning ranks is finite, the actual size of prizes greatly varies across drawings. Lotto is indeed a parimutuel game, with the prizes set equal to a percentage of the total amount bet. A winner at rank \( i = 1, ..., 7 \) will receive an amount equal to \( \alpha_i(1 - \tau)S/x_i \) where \( \alpha_i \) is the share of net-of-tax sales dedicated to winners at rank \( i \), \( \tau \) the lottery tax rate, \( S \) total ticket sales and \( x_i \) the number of winners at rank \( i \). As ticket sales and the number of winners at each rank are random, prize values are random as well. The more

tickets sold, the greater the pie to be divided among winners. The greater the number of winners at a given rank, the smaller the individual shares at this rank. These two factors explain why the actual prize distribution spans a very large set of prizes. Table 2 illustrates this property out of the same dataset by indicating the winners’ minimal and maximal sizes of prize, net of the ticket price of a lottery card (equal to 1.2 euro), for each winning rank.

<table>
<thead>
<tr>
<th>numbers</th>
<th>minimal money prize</th>
<th>maximal money prize</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>107,813</td>
<td>11,000,000</td>
</tr>
<tr>
<td>5+</td>
<td>496</td>
<td>93124</td>
</tr>
<tr>
<td>5</td>
<td>234</td>
<td>2131</td>
</tr>
<tr>
<td>4+</td>
<td>15.2</td>
<td>71.2</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>36.2</td>
</tr>
<tr>
<td>3+</td>
<td>1.2</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Table 2. Size of prizes (net of the ticket price) for each rank in the French Lotto game

The combination of seven winning ranks and a high dispersion of within-rank returns implies that money prizes span a very large spectrum of prizes. In particular most adjacent brackets overlap. Only 0.14% of the whole interval of money prizes between 0 and 11 million euros is not included in one of the seven brackets. Note that for the dataset at hand, the minimal net prize is equal to zero as the minimal prize value is exactly equal to the ticket price. Hence, statistical evidence supports the starting assumption of the next sections which represents the prize payout distribution as a continuous distribution above a non-negative minimal prize.

Table 1 shows a positively skewed prize distribution. A ticket offers large
prizes with long odds. This property will be simply modeled in Section 4 by assuming that the probability density function of money prizes is decreasing with the size of the prize. This assumption may not be uniformly true along the whole interval of prizes due to the parimutuel nature of the game but indisputably captures a central feature of lottery games.

The same statistics can be computed for another lottery game, the Euromillion lottery. Although this type of lottery is less common, it is still worth studying for two reasons. First, it exemplifies a recent trend in which states form consortia for the purpose of offering higher jackpots (Cook and Clotfelter, 1993), a formula which has proved to lure many players. Hence, we may expect multistate lottery games to keep expanding in the future. Second, the prize structure is different from a classic lotto game. It offers a broader range of prizes and much longer odds. It is therefore interesting to check whether the lottery properties that have been outlined are robust to changes in the type of lotteries.

In the Euromillion lottery, five numbers are drawn in the set \( \{1, \ldots, 50\} \) and two bonus numbers are drawn in the set \( \{1, \ldots, 9\} \). This leads to twelve winning ranks defined in table 3. The notation \( n+m \) means that \( n \) numbers are correctly guessed in the first set and \( m \) in the second set. The second column reports the probability of finding the corresponding combination and the next two columns the minimal and maximal money prizes of each rank, net of the ticket price of 2 euros. The statistics are based on the observation of 266 consecutive drawings run between 11-05-2004 and 12-24-2009\(^7\).

\(^7\)Data are taken from http://www.francaise-des-jeux.fr.
Once again, most adjacent intervals of gains overlap. A player may expect as a result to win almost any amount of money between 5 euros and several millions euros in accordance with the result found for the game *Loto*. In both games, a player is however most likely to lose the price of the ticket, with a probability of around $2/3$ in the game *Loto* and $4/5$ in the Euromillion lottery. This introduces a discontinuity in the prize distribution which will be taken into account when the consequences for preferences under risk will be investigated in Section 4. To sum up, three fundamental properties characterize the probability distribution of lotto games: (i) there is a unique level of loss that occurs with a positive probability mass, (ii) the prize payout distribution is best represented by a continuous distribution above a non-negative minimal prize, and (iii) the probability distribution function of money prizes is decreasing.

## 3 Optimal lottery games

In this section a general problem of optimal lottery design is studied. Let us consider the problem faced by a risk-neutral monopoly which sells a lottery card endowed with $n$ payoffs (or prizes) so as to maximize its expected profit.

### Table 3. Probabilities and size of prizes (net of the ticket price) for each rank in the Euromillion lottery

<table>
<thead>
<tr>
<th>numbers</th>
<th>probability</th>
<th>minimum</th>
<th>maximum</th>
<th>numbers</th>
<th>probability</th>
<th>min</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 + 2</td>
<td>$6.55 \times 10^{-5}$</td>
<td>3,000,000</td>
<td>126,231,762</td>
<td>3 + 2</td>
<td>$6.5 \times 10^{-4}$</td>
<td>30.4</td>
<td>130.8</td>
</tr>
<tr>
<td>5 + 1</td>
<td>$9.15 \times 10^{-7}$</td>
<td>80,525</td>
<td>9,839,059</td>
<td>3 + 1</td>
<td>$9.1 \times 10^{-8}$</td>
<td>16.3</td>
<td>37.9</td>
</tr>
<tr>
<td>5</td>
<td>$1.375 \times 10^{-6}$</td>
<td>18,516</td>
<td>918,288</td>
<td>2 + 2</td>
<td>$9.3 \times 10^{-8}$</td>
<td>11.4</td>
<td>36.3</td>
</tr>
<tr>
<td>4 + 2</td>
<td>$1.475 \times 10^{-5}$</td>
<td>670</td>
<td>10,840</td>
<td>2</td>
<td>0.0136</td>
<td>9.5</td>
<td>22.4</td>
</tr>
<tr>
<td>4 + 1</td>
<td>$2.065 \times 10^{-4}$</td>
<td>36</td>
<td>410.5</td>
<td>1 + 2</td>
<td>0.04</td>
<td>4.9</td>
<td>13.4</td>
</tr>
<tr>
<td>4</td>
<td>$3.095 \times 10^{-4}$</td>
<td>17</td>
<td>193.4</td>
<td>2 + 1</td>
<td>0.131</td>
<td>5.2</td>
<td>10.1</td>
</tr>
</tbody>
</table>
A prize $x_i$, $i = 1, ..., n$ is positive if a gain for the purchaser and negative if a cost. Let $\pi_i$, $i = 1, ..., n$ denote the associated cumulative distribution of prizes: $\pi_i = \Pr(x \leq x_i)$. As in Kahneman and Tversky (1992), I assume that players value the gamble $(x_1, \pi_1; ..., x_{k-1}, \pi_{k-1}; x_k, \pi_k; ...; x_n, 1)$, where $x_{k-1} < 0 \leq x_k$, by assigning it the value

$$
\sum_{i=1}^{k-1} [g(\pi_i) - g(\pi_{i-1})]v(x_i) + \sum_{i=k}^{n} [h(1 - \pi_{i-1}) - h(1 - \pi_i)]v(x_i)
$$

where $g(.)$ and $h(.)$ are the probability weighting functions for losses and gains respectively, that map the $[0, 1]$ interval onto itself. The value function $v(.)$ and the probability weighting functions $g(.)$ and $h(.)$ are strictly increasing and twice differentiable over $[-w, \infty]$ and $[0, 1]$ respectively. The value function is not defined over final wealth levels but over gains and losses. This can be justified by the fact that the attractiveness of lottery games do not seem to depend on the level of wealth of participants. The reference point is taken to be the status quo, namely the player’s wealth without gambling. The lower bound $-w$ reflects the maximum loss that a player can undergo. It can be her lifetime wealth or a fraction of it, in case of liquidity constraints for example. Note that such preferences satisfy Cumulative Prospect Theory only by misuse of language, as no restrictions are made on the curvature of the three above-mentioned functions, contrary to Kahneman and Tversky (1992).

Generally speaking, the lottery operator has to decide how many prizes to include in the lottery game, their value and their probability of occurrence. This problem being too complex to tackle, I will consider an alternative operator’s set of choice that will prove to be equivalent to the original one. I assume
that the company chooses which payoffs $x_i$, $i = 1, 2, ..., n$, to associate to a set of cumulative probabilities $\{\pi_1, ..., \pi_n\}$, where the number of payoffs $n$ and all probabilities are exogenously given to the operator. Probabilities are equal to:

$$\pi_i = i/n, \quad i = 1, 2, ..., n$$

In other words, all prizes have the same probability of occurrence $1/n$. The corresponding lottery is denoted $\{x_i : i/n, i = 1, ..., n\}$. Restricting to uniform probability distributions does not loss generality, as a money prize can be offered several times in the same lottery. For instance, a lottery endowed with $q$ prizes $y$ with probability $1/n$ is equivalent to a lottery that offers a single prize $y$ with probability $q/n$. To see this, suppose that the lottery $\{x_i : i/n, i = 1, ..., n\}$ includes a sequence of $q$ same-value prizes $x_{l+1} = x_{l+2} = ... = x_{l+q} = y$ with $1 \leq l < l + q - 1 \leq n$ and $y < 0$ (a similar reasoning would hold with $y \geq 0$). Its value is:

$$... + [g((l + 1)/n) - g(l/n)] v(y) + ... + [g((l + q)/n) - g((l + q - 1)/n)] v(y) + ...$$

After pooling all prizes equal to $y$ and aggregating associated probabilities, it becomes:

$$... + [g((l + q)/n) - g(l/n)] v(y) + ...$$

which is also the value of a lottery offering $y$ with probability $q/n$.

Conversely, an arbitrary lottery where the prize $y$ appears with probability $p$ can be replicated by a lottery with a uniform probability distribution where $y$ appears $np$ times, providing $np$ is an integer. Hence, a strict equivalence holds as long as all probabilities of a lottery can be expressed as multiples of $1/n$. 

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This is a very weak requirement as \( n \) can be raised as high as desired without changing the nature of the prize distribution and its value for a player. For instance, assume that the uniform lottery \( \{ x_i : i/n, i = 1, \ldots, n \} \) replicates an arbitrary lottery in which all probabilities are multiples of \( 1/n \). The expanded uniform lottery \( \{ x'_j : j/rn, j = 1, \ldots, rn \} \) replicates the same lottery where a prize appears \( rq \) times in a row instead of \( q \) times in the initial uniform lottery.

Let us denote \( v_0 \) the reservation value that the consumer obtains by abstaining from gambling. The company chooses prize values which maximizes its profit, while meeting the player’s participation constraint and a set of ordering constraints:

\[
\begin{align*}
\max_{\{x_1, \ldots, x_n\}} & - \frac{1}{n} \sum_{i=1}^{n} x_i \\
\text{s.t.} & \sum_{i=1}^{k-1} [g(i/n) - g((i-1)/n)]v(x_i) \\
& + \sum_{i=k}^{n} [h((n-i+1)/n) - h((n-i)/n)]v(x_i) = v_0 \\
& x_i \geq x_{i-1}, i = 1, \ldots, n \\
& x_0 = -w
\end{align*}
\]

A Lagrange function is formed by appending the objective function and the constraints. The multipliers for the participation constraint, and the ordering constraints \( x_i - x_{i-1} \geq 0 \), are respectively denoted \( \lambda \) and \( \eta_i, \ i = 1, \ldots, n \). A payoff \( x_i \) is optimal if it satisfies the first order conditions (with \( \eta_{n+1} = 0 \)):

\[
\begin{align*}
\frac{g(i/n) - g((i-1)/n)}{1/n + \eta_{i+1} - \eta_i} v'(x_i) &= \frac{1}{\lambda}, \ i = 1, \ldots, k - 1 \\
\frac{h((n-i+1)/n) - h((n-i)/n)}{1/n + \eta_{i+1} - \eta_i} v'(x_i) &= \frac{1}{\lambda}, \ i = k, \ldots, n,
\end{align*}
\]

where \( x_{k-1} < 0 \leq x_k \). The second order condition is:  

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\[ v''(x_i) \leq 0, \; i = 1, \ldots, n. \] (2)

To understand Eq. (1), first assume that the boundary constraints do not bind: \( \eta_{i+1} = \eta_i = 0 \). In this case, the ratio \( [g(i/n) - g((i - 1)/n)]/(1/n) \) (with \( x_i < 0 \)) measures by how much the chance of getting \( x_i \) is distorted by the player. This ratio is equal to one when the probabilities are not transformed. The lottery operator must therefore increase the value of a prize wherever the player’s marginal value \( v'(x_i) \) is "high" or its true probability of occurrence is overestimated by the player (i.e. \( [g(i/n) - g((i - 1)/n)]/(1/n) \) is "high").

This rule offers the best way for the operator to relax the player’s participation constraint and, as a result, to globally reduce the average payout of the lottery.

The multipliers \( \eta_{i+1} \) and \( \eta_i \) reflect the loss of profit associated with the upper boundary constraint \( x_i \leq x_{i+1} \) and the lower boundary constraint \( x_{i-1} \leq x_i \) respectively. If the upper constraint binds at the optimum (\( \eta_{i+1} > 0 \) and \( \eta_i = 0 \)), \( x_i \) is lowered compared to the case \( \eta_{i+1} = \eta_i = 0 \) in order to alleviate the constraint. Conversely, \( x_i \) is greater if the lower boundary constraint binds (\( \eta_i > 0 \) and \( \eta_{i+1} = 0 \)). If the two boundary constraints bind simultaneously (\( \eta_i, \eta_{i+1} > 0 \)), there are two cases. If \( \eta_i > \eta_{i+1} \), alleviating the constraint \( x_{i-1} \leq x_i \) by raising \( x_i \) is more profitable for the firm than reducing the cost of the constraint \( x_i \leq x_{i+1} \) by lowering \( x_i \). The converse is true if \( \eta_i < \eta_{i+1} \).

This section has derived the optimality conditions of a lottery endowed with a finite number of prizes over an exogenous set of probabilities. This problem is reversed in the next section by assuming that the company chooses optimal probabilities over a finite set of fixed payoffs.
4 Implications for preferences under risk

Several properties characterizing the prize payout distribution of lotto games have been outlined in Section 2. Let us recall them: (i) there is a unique level of loss, denoted $y_1$, that occurs with a positive probability mass (equal to $\varphi_1$), (ii) the probability distribution of gains is continuous over the domain $[y, \overline{y}]$ where $y > 0 > y_1$ and (iii) the probability density function is decreasing. In this section, I investigate which type of preferences under risk are consistent with an optimal lottery characterized by such properties. The following assumption formally defines the shape of the optimal game and assumes its existence.

**Assumptions H1.** Let us define a cumulative probability distribution of prizes $F(y) : [-w, \infty[ \rightarrow [0, 1]$ satisfying (i) $F(y) = 0 \forall y < y_1$ and $F(y_1) = \varphi_1$, (ii) $F(y)$ is strictly increasing and twice differentiable over $[y, \overline{y}]$ with $y > 0 > y_1$, $F(y_1) = \varphi_1$, $F(y) < 1 \forall y < \overline{y}$, $F(\overline{y}) = 1$ and (iii) $F''(y) < 0 \forall y \in [y, \overline{y}]$. Let us define the prize function $y(\pi) = F^{-1}(\pi) : [\varphi_1, 1] \rightarrow [y, \overline{y}]$. It is further assumed that $F(y)$ maximizes the company’s profit, exists, and brings about a non-negative expected profit to the company: $-\varphi_1 y_1 - \int_{\varphi_1}^{1} y(\pi)d\pi \geq 0$.

As in the previous section, I first assume that the company chooses optimal payoffs over a discrete grid of probabilities $\pi_i = i/n$, $i = 1, 2, ..., n$. The optimality conditions related to this problem have been presented in the previous section. Then I look how these optimality conditions are affected when I turn to the kind of distribution hypothesized in H1. To do so, the discrete uniform probability $\{x_i : i/n, i = 1, ..., n\}$ is restricted in the following way: (i) the first
$n \varphi_1$ prizes $x_i$ takes the value $y_1$, (ii) each subsequent prize satisfies $x_i = y(i/n)$ \forall i = n \varphi_1 + 1, \ldots, n$, and (iii) $n \to \infty$. Proposition 1 follows:

**Proposition 1.** Under H1:

(a) $v''(y_1) \leq 0$ and $v''(y) \leq 0 \ \forall y \in [y, \bar{y}]$,

(b) $h''(\pi) \leq 0$ over $\pi \in [0, 1 - \varphi_1]$.

(c) $\int_0^{\varphi_1} g''(\pi)d\pi \leq 0$, if the wealth constraint is not binding ($y_1 > -w$),

(c') $\int_0^{\varphi_1} g''(\pi)d\pi \leq 0$, if the wealth constraint is binding ($y_1 = -w$).

Proposition 1(a) comes from the second order conditions of the lottery problem. It states that the value function must be locally concave around the ticket price$^8$, and globally concave over the domain of positive prizes. The concavity of the value function makes unappealing to players the inclusion in the lottery game of ever increasing money prizes, unless they attach high enough probability weights to large prizes. The inspection of the first order condition of the lottery problem over the continuous part of the distribution shows that such an offsetting effect occurs, where $\lambda$ is the Lagrange multiplier of the problem (see the proof of Proposition 1(b)):

$$h'(1 - \pi)v'(y(\pi)) = 1/\lambda, \ \pi \in [\varphi_1, 1].$$

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$^8$This result is partially driven by the assumption that the value function is everywhere twice differentiable. Analyzing the possibility supported by experimental evidence that the value function is steeper for losses than for gains would need another set of assumptions (a reference point equal to the wealth minus the ticket price and a kinked value function around the reference point) which is beyond the scope of this paper.
For the lottery to be endowed with a continuum of prizes necessitates that the marginal probability weights attached to larger prizes are increasing due to the decreasing profile of the marginal value function. This explains the convexity of the probability weighting function in the domain of gain and for small probabilities stated in 1(b). Hence, propositions 1(a) and 1(b) are necessary for money prizes to be distributed over a continuous interval.

Proposition 1(c) shows that the weighting function is concave in average in the domain of loss for small probabilities if the wealth constraint is not binding and undetermined in the contrary case. This constraint on preferences leads the operator to offer a unique level of loss, the ticket price, with a positive probability mass. Result (c′) shows that the average concavity of \( g(.) \) may be lost if the operator were willing to set a ticket price below the maximum loss that a player may incur.

The next proposition exploits the third stylized fact, namely a decreasing money prize probability distribution function.

**Proposition 2.** Let us define an index of curvature for the probability weighting function \( \gamma(\pi) = h''(1 - \pi)/h'(1 - \pi) \) and one for the value function : \( \sigma(y) = v''(y)/v'(y) \). The assumption \( F''(y) < 0 \) in H1 implies

\[
\gamma'(\pi) \left[ \gamma(\pi) \right]^2 > \sigma'(y(\pi)) \left[ \sigma(y(\pi)) \right]^2, \quad \pi \in [\varphi_1, 1].
\]

Proposition 2 shows that the index of curvature of the probability weighting function must increase faster (or decrease slower) than the same index for the value function for the money prize probability distribution function to decrease.
This is overall a rather weak restriction on risk preferences.

5 Conclusion

This article investigates which forms of preferences under risk are consistent with profit-maximizing lotto games. The approach employed by Friedman and Savage (1948) is extended by introducing a reference point against which changes in wealth are valued and probability transformation. Both alterations make the model directly comparable to experimental results found in the context of Cumulative Prospect Theory. By exploiting some basic stylized facts that define the general shape of lotto games, a number of meaningful restrictions on preference are found. The value function is globally concave over the domain of positive prizes offered by the lottery, and the probability weighting function is concave for low probabilities in the gain domain and concave in average for low probabilities in the loss domain. Remarkably, all these properties match the corresponding results found in the experimental literature.

Some extensions are left for future investigations. The model takes for granted the existence of optimal lottery games with realistic features and exhibits a few constraints on preferences under risk. It would be interesting to reverse the article’s perspective by studying under which circumstances non-expected utility theories such that Cumulative Prospect Theory lead profit-maximizing operators to sell lottery games that resemble real world ones. The theoretical framework could also be enriched. The present model is static and assumes homogeneous preferences through the artefact of a representative player. It would be worth studying to what extent the findings of the model are modi-
fied when repeated play is allowed or when some heterogeneity in preferences is taken into account by the operator.

References


Appendix

Proof of Proposition 1.

The lottery defined in H1 can be replicated by a uniform probability distribution \( \{x_i : i/n, i = 1, \ldots, n\} \), in which (i) the first \( n\varphi_1 \) prizes \( x_i \) takes the value \( y_1 \), (ii) each subsequent prize satisfies \( x_i = y(i/n) \forall i = n\varphi_1 + 1, \ldots, n \), where \( y(\pi) \) is defined in H1, and (iii) \( n \to \infty \).

(a) All \( x_i = y_1 \) must meet the same second order condition (2) \( v''(x_i) \leq 0 \), \( i = 1, \ldots, n\varphi_1 \) or equivalently \( v''(y_1) \leq 0 \). For \( x_i \in [y, \bar{y}] \), the second order condition (2) \( v''(x_i) \leq 0 \), \( i = n\varphi_1 + 1, \ldots, n \) becomes \( v''(y) \leq 0 \) \( \forall y \in [y, \bar{y}] \) once \( n \to \infty \).

(b) Over the continuous part of the prize distribution, the first order conditions (1) hold with \( \eta_i = 0 \) as \( x_i = y(i/n) > y((i - 1)/n) = x_{i-1} \), \( i = n\varphi_1 + 1, \ldots, n \):

\[
\frac{h(1 - i/n + 1/n) - h(1 - i/n)}{1/n} v'(x_i) = 1/\lambda, \quad n\varphi_1 + 1, \ldots, n.
\]

When \( n \to \infty \), the discrete set of probabilities \( \{i/n, i = 1, \ldots, n\} \) turns into a continuum of probabilities \( \pi \) over \( [0, 1] \). The first order condition expresses as:

\[
h'(1 - \pi) v'(y(\pi)) = 1/\lambda, \quad \pi \in [\varphi_1, 1]. \tag{4}
\]

Similarly, let us take the derivative of Eq. (4) with respect to \( \pi \). After rearranging the terms:

\[
h''(1 - \pi) = \frac{v''(y(\pi))}{v'(y(\pi))} h'(1 - \pi) y'(\pi), \quad \pi \in [\varphi_1, 1]. \tag{5}
\]

Since \( v(y) \), \( h(1 - \pi) \) and \( y(\pi) \) are strictly increasing functions over their respective intervals \([-w, \infty[ \), \( [0, 1] \) and \( [\varphi_1, 1] \), \( v''(y) \leq 0 \) over \( [y_1, \bar{y}] \) implies \( h''(1 - \pi) \leq 0 \) over \( \pi \in [\varphi_1, 1] \) or \( h''(\pi) \leq 0 \) over \( \pi \in ]0, 1 - \varphi_1[ \).
The first order conditions for \( x_i = y_1, \ i = 1, ..., n \varphi_1 \), are given by the \( n \varphi_1 \) conditions (1). For \( i = 1 \) and \( i = n \varphi_1 \), these conditions can be written as:

\[
\begin{align*}
\eta_2 - \eta_1 &= \lambda u'(x_1) g(1/n) - 1/n \\
\eta_{n \varphi_1 + 1} - \eta_{n \varphi_1} &= \lambda u'(x_{n \varphi_1}) \left[ g(\varphi_1) - g(\varphi_1 - 1/n) \right] - 1/n.
\end{align*}
\]

The multiplier \( \eta_{n \varphi_1 + 1} \) is equal to 0 as \( y_1 = x_{n \varphi_1} < x_{n \varphi_1 + 1} = y \). If the wealth constraint is not binding, i.e. \( y_1 > -w \), \( x_1 > x_0 \) and hence \( \eta_1 = 0 \).

Combining these two equations and replacing \( x_i, i = 1, n \varphi_1 \), by \( y_1 \):

\[
\frac{g(1/n)}{1/n + \eta_2} = \frac{g(\varphi_1) - g(\varphi_1 - 1/n)}{1/n - \eta_{n \varphi_1}}.
\]

This equality can be rewritten as:

\[
\frac{1/n - \eta_{n \varphi_1}}{1/n + \eta_2} = \frac{g(\varphi_1) - g(\varphi_1 - 1/n)}{1/n} \leq 1
\]

since \( -\eta_{n \varphi_1} \leq \eta_2 \) (all multipliers are non-negative in the lottery problem).

Taking \( n \to \infty \):

\[
g'(\varphi_1) - g'(0) \leq 0.
\]

The inequality in (c) becomes apparent after integrating \( g''(\varphi) \) over \([0, \varphi_1]\).

(c') If the wealth constraint is binding: \( y_1 = -w, \eta_1 > 0 \). Eq. (6) becomes:

\[
\frac{g(1/n)}{1/n + \eta_2 - \eta_1} = \frac{g(\varphi_1) - g(\varphi_1 - 1/n)}{1/n - \eta_{n \varphi_1}}.
\]

This equality can be rewritten as:

\[
\frac{1/n - \eta_{n \varphi_1}}{1/n + \eta_2 - \eta_1} = \frac{g(\varphi_1) - g(\varphi_1 - 1/n)/g(1/n)}{1/n}.
\]

Taking the limit \( n \to \infty \):

\[
\frac{1/n - \eta_{n \varphi_1 - 1}}{1/n + \eta_1 - \eta_0} = \frac{g'(\varphi_1)}{g'(0)} \begin{cases} 
\leq 1 & \text{if } \eta_1 \leq \eta_2 + \eta_{n \varphi_1} \\
> 1 & \text{if } \eta_1 > \eta_2 + \eta_{n \varphi_1}.
\end{cases}
\]

Hence, the binding of the wealth constraint may reverse the properties of the weighting function between 0 and \( \varphi_1 \) if the wealth constraint is costly enough for the firm, that is, if \( \eta_0 \) is relatively high compared to \( \eta_{n \varphi_1 - 1} + \eta_1 \). □
Proof of Proposition 2. Equation (5) in the Proof of Proposition 1 can be rewritten as:

\[ y'(\pi) = \gamma(\pi)/\sigma(y), \pi \in [\varphi_1, 1] \]

with \( \gamma(\pi) = h''(1 - \pi)/h'(1 - \pi) \) and \( \sigma(y) = v''(y(\pi))/v'(y(\pi)) \). \( y(\pi) \) is the inverse of the cumulative distribution function over the continuous interval of prizes: \( y(\pi) = F^{-1}(\pi), \pi \in [\varphi_1, 1] \). Hence, \( F''(y) < 0, y \in [\underline{y}, \overline{y}] \) is equivalent to \( y''(\pi) > 0, \pi \in [\varphi_1, 1] \) or:

\[ \gamma'(\pi)\sigma(y(\pi)) - \sigma'(y(\pi))y'(\pi)\gamma(\pi) > 0, \pi \in [\varphi_1, 1]. \]

Rearranging the terms:

\[ \frac{\gamma'(\pi)}{|\gamma(\pi)|^2} > \frac{\sigma'(y(\pi))}{|\sigma(y(\pi))|^2}, \pi \in [\varphi_1, 1], \]

which completes the proof. \( \square \)