Monetary Rules in a Two-Sector Endogenous Growth Model

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Abstract: We study a two-sector endogenous growth model with physical capital and human capital accumulation. The co-existence of two capital goods allows the economy to display unbounded growth without relying upon external effects. The demand of money is motivated on the ground of a fractional cash-in-advance constraint on consumption expenditures. We consider, in sequence, two monetary rules implemented by the Central Bank. First, we assume that the latter pegs the money growth rate and then the nominal interest rate according to a Taylor feedback rule. When the Central Bank pegs the money growth rate there exists a unique balanced growth path which turns out to be indeterminate for a low amplitude of the liquidity constraint and/or for a low enough intertemporal elasticity of substitution in consumption. On the other hand, when monetary policy is conducted according to a feedback Taylor rule, a liquidity trap equilibrium may coexist with an interior Leeper equilibrium and both solutions are locally determinate. As a consequence, the implemented equilibrium will be selected on the ground of agents’ expectations.

Keywords: Cash-in-Advance; Two-Sector; Endogenous Growth, Monetary Policy.

JEL Classification Numbers: E32, E52, O41, O42.


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1 Introduction

It is well known that when one introduces positively valued fiat money in infinite horizon economies, the stability features of the stationary solutions may be richer than those characterizing the non-monetary environments. The introduction of money represents indeed a friction in the capital market, since transactions now require a specific medium of exchange whose real value is determined by the price level and thus by the inflation path which in turns affects the whole structure of the relative prices. It is also well known that the equilibrium features of the economy in terms of existence, stability and optimality depend upon the monetary policy implemented by the Central Bank; as a matter of fact, results are not the same when one assumes a monetary policy consisting in pegging the money growth rate or in pegging the nominal interest rate on the ground of a specific feedback Taylor rule.

What has been less analyzed, however, is the role played by the different monetary policies on the features of economies displaying unbounded growth. In such a case, indeed, one may wonder how the different monetary rules do influence the number of the balanced growth paths as well and their stability. To provide some answers to these questions, in this paper we study an infinite-horizon discrete-time economy populated by a representative agent whose preferences are defined over consumption and accumulating two capital goods, respectively a physical capital good and a human capital good, safe government bonds and money balances. The demand of money is motivated by the presence of a fractional cash-in-advance constraint on consumption expenditures which compels agents to buy a fraction of the value of the consumption good out of the money balances available in the foregoing period. As a consequence, consumption requires a pre-investment in money balances entailing an opportunity cost represented by the nominal interest rate. The assumption of a fractional cash-in-advance constraint allows us to account for a variable degree of financial market imperfection or, equivalently, for a variable velocity of circulation of money representing simply the inverse of the amplitude of the liquidity constraint, according to the Cambridge Balance Approach.

We assume a two-sector model with two productive inputs: physical capital and human capital. The first sector is devoted to the production of the consumption good and of the physical capital good, meanwhile the human capital is produced according to a different technology. The presence of two capital goods allows the economy to display unbounded growth without relying on the presence of external effects in production, as in Romer (1986) or on the presence of productive public spending, as it is emphasized in Barro (1990). Government issues bonds and levies taxes to finance public expenditures and the Central Bank operates in the bond market by issuing unbaked fiat money. We focus on two different monetary policies implemented by the Central Bank: according to the first, it pegs the money growth rate, meanwhile, according to the second, it pegs the nominal interest rate on the ground of a feedback Taylor rule. We will show that the choice of the monetary policy implemented entails dramatic consequence on the number of the balanced growth paths as well as on their stability features.

Analyzing a two-sector model with cash-in-advance on consumption expenditures is by far more than a mere theoretical curiosity. It is not a mystery, indeed, that the production of consumption goods is usually made possible by means of technologies which are different with respect to the ones employed to produce the investment goods, as many empirical studies suggest. As an example, according to Takahashi et al. (2012) and Baxter (1996) empirical estimates, the consumption sector appears to be more capital intensive that the investment one. In addition, the two-sector models, even en absence of some market imperfection, are known to display cyclical behavior as it is shown, among the others, in Benhabib and Nishimura (1985); it is then worthwhile to asses how such cycles do persist and even become more
pervasive when one relies on some monetary extension of the basic framework. Eventually, by extending the hypothesis of different sectoral technologies to an endogenous growth model, one is able to further enrich the dynamic features since, in addition to the typical mechanism relying on the asymmetric factor intensities, there are those linked to the growth engine.

The literature on the subject here treated, may be classified into two subsets. In a first one, there are the continuous-time models. Bond et al. (1996), as an example, consider an optimal two-sector endogenous growth model with two capital goods and show that the model exhibits the typical saddle-path stability, whatever the factor intensity across the sectors. Brito and Venditti (2010), on the other hand, extend the framework of Bond et al. (1996), by accounting for economy-wide external effects. Assuming decreasing returns to scale at the private level and constant returns to scale at the social level, they show that local and global indeterminacy may arise for any value for the elasticity of intertemporal substitution in consumption and any sign for the capital intensity difference across sectors.

In a second strand of literature, there are the discrete-time models. Mino et al. (2008) examine the dynamics properties of an endogenously growing economy with sector-specific spillover effects. They find that conventional results obtained in continuous time frameworks do not more hold. In particular, they show that usual necessary and sufficient conditions in terms of factor intensities for local determinacy are no more true, whereas the magnitudes of time preference and capital depreciation rates play essential roles. Drugeon (2013) provides conditions for optimal cycles within constant returns to scale technologies while Takahashi (2008) extends the analysis to multi-sector economies. Mino (1997) considers a two-sector model with a consumable physical capital good and a human capital good and cash-in-advance constraint on consumption and investment in physical capital. He shows that there may emerge two steady states; these results depend crucially on the hypothesis that the cash-in-advance constraint applies on the whole consumable physical capital good, in contrasts to our economy where only consumption expenditures require previous investment in money balances.

Another bunch of the literature considers the effects of monetary rules on the stability properties of models displaying exogenous growth. Bosi et al. (2005b) and Bosi et al. (2005a) study the effects of a monetary policy consisting in pegging the money growth rate on local determinacy with fractional liquidity constraint on consumption in a pure exchange economy and in a two sector model, respectively, and find that when the amplitude of the liquidity constraint decreases, the scope for local indeterminacy improves. On the other side, Benhabib et al. (2001) and Le Riche et al. (2017) prove the existence of multiple equilibria and analyze their stability properties under the hypothesis that the Central Bank implements a Taylor rule. Benhabib et al. (2001), in a pure exchange economy where money enters the utility function, find that the Taylor equilibrium is locally unique but an unintended stable liquidity trap equilibrium may emerge. Le Riche et al. (2017) consider a productive economy without capital accumulation and with a partial cash-in-advance constraint. Here, the stability of the Taylor equilibrium and of the liquidity trap equilibrium depends dramatically upon the amplitude of the financial constraint.

Within the two monetary policies we consider, we carry out a complete steady state analysis and show that in both frameworks the balanced growth paths are locally unique. Such a result should be the natural consequence of the absence of external effects in production and of the fact that the sole market imperfection relies on the presence of the cash-in-advance constraint. On the other hand, when the Central Bank pegs the money growth rate, the uniqueness of the balanced growth path is even global, meanwhile when the monetary policy is conducted by implementing a nominal interest rate feedback rule, two stationary solutions from a global perspective may arise: the first one is associated to a positive
interest rate, the second corresponding to its zero lower bound, i.e. the liquidity trap. However, all the stationary solutions are characterized by the same growth rate for the real variables, no matter what the monetary policy implemented is, since the former depends uniquely upon the features of the production functions.

The stability properties of the balanced growth paths differ dramatically according to the monetary policy rule implemented by the Central Bank. Under the hypothesis that the latter pegs the money growth rate, we show that the unique balanced growth path becomes locally indeterminate as soon as the amplitude of the financial constraint is set lower and lower and/or there are strong enough income effects in intertemporal consumption. Such results are at first sight counter-intuitive since it is commonly thought that the decrease of the market imperfection should reduce the scope for indeterminacy; however, they confirm the findings in Bosi et al. (2005a) according to which a relaxation of the financial constraint improves the scope for indeterminacy. The mechanism leading to indeterminacy is to be viewed in the reaction of the current nominal interest rate with respect to an anticipated increase in the future one. If the amplitude of the liquidity constraint is large enough, faced with an increase in the expected nominal interest rate, agents will react by decreasing the money holding, provoking thus a small increase in the current nominal interest rate. On the other hand, if the share of consumption to be bought cash is low enough, the diminution of money holding will be only slight, provoking thus an over-reaction of the current nominal interest rate which will thus undertake a converging oscillatory path. Notice that the mechanism described is invariant in respect to the assumption concerning the relative capital intensities in the two productive sectors, and thus the technological side of the economy turns out to be neutral in terms of the stability properties of the balanced growth path.

When the Central Bank follows a feedback Taylor rule, both the liquidity trap equilibrium and the Leeper interior equilibrium are locally determinate, whatever the parameter configuration is. This is the direct consequence of the fact that, since the Taylor rule links the nominal interest rate to the deflation rate, the dynamic system looses one dimension and thus the amplitude of the financial constraint no longer affects neither the balanced growth path nor the transitional dynamics around it. According to our finding, hence, even under the hypothesis that the monetary policy is conducted on the ground of a feedback Taylor rule, neither the relative capital intensities across sectors nor the degree of aggressiveness of the monetary policy do entail relevant consequences on the qualitative feature of the local dynamics, since determinacy of equilibrium is always bound to prevail.

The remainder of the paper is organized as follows. In Section 2 we present the economy; we describe the fiscal policy of the government, the monetary policy pursued by the Central Bank, the household’s behavior and the firms’ technology. Section 3 is devoted to the analysis of the economy under the hypothesis that the Central Bank pegs a constant growth money rate meanwhile Section 4 is devoted to the analysis of the economy under the hypothesis that the Central Bank follows a Taylor rule. Some concluding remarks are left to Section 5 and the main proofs of the propositions are gathered in the Appendix.

2 The model

We consider an infinite horizon discrete time economy populated by the government, the Central Bank, a large number of identical infinitely lived households and two representative firms. In the sequel, we will describe the government and the Central Bank policies, the household behavior and the technology available to the firms allowing the latter to produce the two final goods.
2.1 The Government and the Central Bank

The government in each period $t$ issues nominal bonds, denoted by $B^G_{t+1}$, in order to reimburse previous debt $B^G_t$ and finance its nominal secondary deficit, defined as the excess of payment of interests, $i_t B^G_t$, where $i_t$ is the nominal interest rate, plus nominal public spending, $\tilde{G}_t$, on nominal taxation, denoted by $T_t$. In period 0 the initial nominal debt is $B^G_0$. All through the paper, we will assume that the demand of public spending $\tilde{G}_t$ is addressed to the sector producing, as we will explain in the sequel, the consumable physical capital. This hypothesis is easily justified on the ground that public spending can be fully assimilated to the corresponding consumption purchases effectuated by the households. Hence we have that nominal bonds evolve through times according to the following law of motion:

$$B^G_{t+1} = (1 + i_t)B^G_t + \tilde{G}_t - T_t.$$  \hspace{1cm} (1)

The Central Bank, on the other hand, issues money against the purchase of government bonds through open market operations. Denoting $B^{CB}_{t+1}$ the amount of nominal government bonds purchased by the Central Bank in period $t$, $M^S_{t+1}$ the stock of nominal balances supplied to the economy at the outset of period $t$, and $\tau_t$ the profits corresponding to the money creation, the dynamic budget constraint of the Central Bank writes:

$$B^{CB}_{t+1} + \tau_t = (1 + i_t)B^{CB}_t + M^S_{t+1} - M^S_t.$$  \hspace{1cm} (2)

Notice that the Central Bank in each period creates or withdraws money as a counterpart of its purchases or sales of government bonds. It follows that the Central Bank operates in the financial market in order to attain its objectives, which may consist either in the control of the money growth rate or in the that of the nominal interest rate. We first assume that the Central Bank pegs the money growth rate, then we will turn the attention to the case where the Central Banks modifies continuously the nominal interest rate in response to the possible gaps of the realized inflation rate from the chosen target, according to an opportune Taylor rule. When the Central Bank pegs the money growth rate, denoting the total supply of money in period $t$ by $M^S_t$ and the constant rate of money creation $\hat{\gamma}$, we have that the supply of money balances satisfies:

$$M^S_t = (1 + \hat{\gamma})M^S_{t-1} = \gamma M^S_{t-1}.$$  \hspace{1cm} (3)

where $\gamma = 1 + \hat{\gamma}$ is the constant money growth factor. Let $\tau_t$ be the profits of the Central Bank in period $t$, which represent the money creation or, if negative, money withdrawal; they are paid to or withdrawn from the households in the same period by means of lump sum transfers or taxes. Such profits, in period $t$, are given by

$$\tau_t = \hat{\gamma}M^S_{t-1}.$$  \hspace{1cm} (4)

As a consequence, in period $t$, the total supply of money $M^S_t$ satisfies $M^S_t = \gamma^t M^S_0$, where $M^S_{t=0} = M^S_0$ is the initial amount of money balances available in period zero. In Section 4 we will consider, by contrast, the case where the Central Bank pegs the nominal interest rate following a Taylor rule: in such a case the growth rate of money becomes endogenous, since it reacts as a function of the existing gap between the realized inflation rate and the chosen target, by mean of a depreciation or appreciation of the nominal interest rate.
2.2 Technology

In the economy there are two representative firms, each of them operating in a specific sector and producing one particular good. The first firm produces a consumable physical capital good, denoted by $Y^T_t$, meanwhile the second representative firm is devoted to the production of another capital good, denoted by $Y^N_t$, which from now on we will refer as to "human" capital. In the sequel, we will assume the consumable physical capital good $Y^T_t$ to be taken as the numéraire, so that all the relative prices will be expressed in terms of units of the former. Each of the two firms has access to a specific constant returns to scale technology which uses the two capital goods as productive inputs: the physical capital $K_j$, which is produced by the first firm, and the human capital, $H_j$, which represents the productive outcome of the second firm.

Since markets are perfectly competitive and there are not any barriers on the circulation of the goods, both productive factors turn out to be perfectly mobile across sectors. Notice that we do not include labor among the productive factors, since we assume that it is embedded in the human capital, which can be then viewed as the measure of the effective units of labor, keeping its supply identically equal to one.

The technology allowing the production of human capital can thus be interpreted as reflecting the accumulation of education and skills, accumulation which is converted into human capital by an appropriate combination of the amount of physical and human capital. The constant returns to scale technologies producing, respectively, the consumable physical capital good $Y^T_t$ and the human capital good $Y^N_t$ are thus denoted as follows:

$$Y^T_t = F^T(K^T_j, H^T_j), \quad Y^N_t = F^N(K^N_j, H^N_j) \tag{5}$$

where $K^j_j, H^j_j, j = T, N$, denote, respectively, the amount of the physical capital and of the human capital employed in the $j$ sector. The above defined production functions satisfy the following standard assumptions:

**Assumption 1** The production functions $F^j : \mathbb{R}^2_+ \to \mathbb{R}^2_+$, $j = T, N$, are $C^3$, increasing, concave, homogeneous of degree one and satisfy the Inada conditions $\lim_{K^j \to 0} F^j_K = +\infty$, $\lim_{H^j \to 0} F^j_H = +\infty$, $\lim_{K^j \to +\infty} F^j_K = 0$, $\lim_{H^j \to +\infty} F^j_H = 0$.\(^1\)

The full employment conditions require thus that the produced amount $Y^T_t$ of the consumable physical good is devoted to satisfy the demand of investment in the physical capital, of households’ consumption and of public spending, meanwhile the production $Y^N_t$ of the human capital good is completely converted into investment in human capital. Formally, we have:

$$Y^T_t = K_{t+1} - (1 - \delta^T)K_t + C_t + G_t, \quad Y^N_t = H_{t+1} - (1 - \delta^N)H_t \tag{6}$$

where $K_{t+1}$ is the physical capital available in period $t + 1$, $\delta^T \in (0, 1)$ the depreciation rate of physical capital, $C_t$ the share of the good $Y^T_t$ that is consumed by households, $G_t$ the share that is consumed by the government, $H_{t+1}$ the human capital available at period $t + 1$, and $\delta^N \in (0, 1)$ the depreciation rate of human capital. Each of the two firms operates in a perfectly competitive market and thus takes the prices of its produced good as well as those of the productive inputs as given. To this end, let us denote $r_t$ the real rental rate of physical capital $K_t$ in terms of the good $Y^T_t$, $w_t$ the real rental rate of human capital $H_t$.

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\(^1\) $F^j_K(K^j, H^j)$ and $F^j_H(K^j, H^j)$ denote, respectively, the derivatives $\partial F^j(K^j, H^j)/\partial K^j$ and $\partial F^j(K^j, H^j)/\partial H^j$.  

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5
in terms of the good \( Y_t^N \) and \( p_t \) the price of the human capital good in terms of the physical consumable capital good: all these prices are given from the point of view of each firm who must conversely choose the quantities of physical and human capital to employ in order to maximize her profits. As a matter of fact, the two firms solve, respectively, the following maximization programs:

\[
\max_{\{k_t^N, h_t^N\}} p_t F^N(K_t^N, H_t^N) - r_t k_t^N - p_t w_t H_t^N.
\]

As usual, at the optimum, the marginal productiveness of each input must equalize the unitary cost of the latter. Taking into account the definitions of the relative prices previously introduced, we have that the first-order conditions relative to period \( t \) of the profit maximization in each sector and for each productive input are immediately derived as

\[
\begin{align*}
    r_t &= F^T(K_t^T, H_t^T) = p_t F^N(K_t^N, H_t^N) \\
    \omega_t &= p_t w_t = F^T(K_t^T, H_t^T) = p_t F^N(K_t^N, H_t^N)
\end{align*}
\]

where we have denoted \( \omega_t \) the real rental rate of human capital in terms of units of the consumable good. Since both firms employ the same productive factors and the full employment condition do hold, we can express the two production functions in terms of the shares \( s \) and \( u \), respectively, of physical capital and human capital devoted to the production of the good \( Y_t^T \). In addition, the constant returns to scale hypothesis allows us to express the production functions in their intensive forms. To this end, setting \( k_t^T = K_t^T / H_t^T = s_t K_t / u_t H_t \) the physical capital to human capital ratio in the physical capital sector and \( k_t^N = K_t^N / H_t^N = [(1 - s_t)K_t]/[(1 - u_t)H_t] \) the physical to human capital ratio in the human capital sector, the production of the consumable physical capital good \( Y_t^T \) and of the human capital good \( Y_t^N \) can be thus written as:

\[
\begin{align*}
    Y_t^T &= F^T(s_t K_t, u_t H_t) = u_t H_t f^T(k_t^T), \quad Y_t^N = F^N((1 - s_t)K_t, (1 - u_t)H_t) = (1 - u_t)H_t f^N(k_t^N).
\end{align*}
\]

Straightforward computations yield the following expressions of the shares \( s \) and \( u \), respectively, of physical capital and human capital devoted to the production of the good \( Y_t^T \) in terms of the differences in capital intensities in the two sectors:

\[
\begin{align*}
    u_t &= \frac{k_t - k_t^N}{k_t^T - k_t^N} \quad \text{and} \quad s_t = \frac{k_t^N (k_t - k_t^N)}{k_t^T (k_t^T - k_t^N)}
\end{align*}
\]

where \( k_t = K_t / H_t \) is the ratio of physical to human capital. The following Lemma is an application of the Stolper-Samuelson \((dr/dp, dw/dp)\) effect and the Rybznyski \((dy^T/dk, dy^N/dk)\) one and it is useful in order to derive the expression of the derivatives of the relative rental prices as well as to show that \( \text{sign}(dw/dp) = \text{sign}(d\omega dp) \).
Lemma 1. Under Assumption 1, the following relationships hold:

\[
\frac{dr}{dp} = \frac{dy}{N} \frac{dk}{(k^N - k^T)} \quad \frac{dy}{N} \frac{d\omega}{dp} = \frac{dy}{N} \frac{dy}{(k^N - k^T)} = -\frac{k^T}{(k^N - k^T)} \\
\text{and} \\
\frac{dr}{dp} = f''(k^T) \frac{f'''(k^N)}{(k^N - k^T)f''(k^T)}, \quad \frac{dy}{dt} = -k^N \frac{f''(k^N)}{(k^N - k^T)f''(k^N)}.
\]

(12)

**Proof:** See Appendix 6.1.

Notice that in the above Lemma we find the well-known duality between the Rybczynski effect and Stolper-Samuelson one. The sign of these effects depends upon the relative capital intensity difference $k^N - k^T$ across sectors. The sign of $k^N - k^T$ is positive (resp. negative) if and only if the consumable capital good is human capital (resp. physical capital) intensive. Under a human capital (resp. physical capital) intensive consumption good the Stolper-Samuelson effect claims that an increase (resp. decrease) of the relative price decreases (resp. increases) the rental rate of capital and raises (resp. decreases) the rental rate of human capital, whereas the Rybczynski effect claims that an increase (resp. decrease) of the physical capital-human capital ratio decreases (resp. increases) the production of the consumable physical capital good and increases (resp. decreases) the production of the human capital good.

### 2.3 Households

We consider an infinite horizon discrete time economy populated by a constant mass of agents whose size is normalized to one. The preferences of the representative agent are described by the following intertemporal utility function:

\[
\sum_{t=0}^{\infty} \beta^t \frac{C^{1-\sigma}}{1-\sigma}
\]

where $\sigma > 0$ denotes the inverse of the elasticity of intertemporal substitution in consumption and $\beta \in (0, 1)$ the discount factor.\(^2\) Notice that the homothetic functional form of the utility defined in (13) is compatible with a balanced growth path for all the relevant real variables. As usual in infinite horizon growth models, the degree of patience of the agents must be sufficiently low when compared to the rate of growth of the economy, in order to ensure that even when time goes to infinity, the present value of the marginal utility converges to zero and no advantageous arbitrage are possible in order to increment total utility by diverting GDP from investment to consumption. To this end, the next assumption puts an upper threshold for the discount factor such that, for all the discount factors below it, the transversality conditions are respected. More in details, let denote $g_{\text{max}}$ the maximal attainable consumption growth factor among all the feasible paths. Then we assume the following:

**Assumption 2** The maximal attainable consumption growth factor $g_{\text{max}}$ satisfies the following condition:

\[
\beta < \frac{1}{(g_{\text{max}})^{1-\sigma}}
\]

\(^2\sigma = 1\) corresponds to the logarithm case.
In each period, agents consume, pay taxes, invest in both capital goods, in government bonds, and in money balances. They finance their expenditures out of capital income, nominal government bond held at the outset of the period, inclusive of the nominal interest rate, and of money balances, inclusive of the profits that the Central Bank distributes to them. It follows that when maximizing (13), agents must respect the dynamic budget constraint

\[
B_{t+1} + p^T_t C_t + p^T_t K_{t+1} + p^N_t H_{t+1} + M_{t+1} + T_t =
(1 + i_t)B_t + M_t + \tau_t + p^T_t r_t K_t + p^T_t (1 - \delta^T)K_t + p^N_t w_t H_t + p^N_t (1 - \delta^N)H_t
\]

where \(B_t\) denotes the safe nominal bonds held by the representative agent, \(\tau_t\) Central Bank’s profits distributed to the agents, and all the other variables have already been defined. The initial endowment of bonds held in period zero by the representative household is denoted \(B_0\) and the initial amount of his nominal balances in period zero is denoted \(M_0\). We assume in addition that agents are subject to a fractional cash-in-advance (CIA) constraint on their consumption purchases, following analogous lines as in Hahn and Solow (1995):

\[
\psi p^T_t C_t \leq M_t. \tag{15}
\]

Such a constraint claims that at least a share \(\psi \in (0, 1]\) of consumption expenditures must be financed out of money balances \(M_t\) held at the outset of the period in which consumption takes place. Notice that under a binding CIA constraint the share \(\psi\) can be viewed as a proxy of the inverse of the velocity of circulation of money. Actually, the velocity of circulation of money is usually defined by the number of time that money passes from one hand to another in order to finance the transactions of the whole \(GDP\), whose variation not necessary converts into an analogous variation of consumption expenditures. However, as soon as the comovements of consumption and \(GDP\) are not too much diverging, the inverse of our parameter \(\psi\) can be viewed as a good approximation of the velocity of circulation of money. Our formulation thus allows to study how the features of the economy do change as soon as the velocity of circulation of money is made to vary. Choosing again the consumption good as the \(numéraire\), we can write the budget constraint and the CIA constraint in real terms:

\[
q_t B_{t+1} + C_t + K_{t+1} + p^N_t M_{t+1} + q_t T_t
= q_t(1 + i_t)B_t + q_t M_t + q_t \tau_t + r_t K_t + (1 - \delta^T)K_t + p^N_t w_t H_t + p^N_t (1 - \delta^N)H_t \tag{16}
\]

and

\[
\psi C_t \leq q_t M_t \tag{17}
\]

where \(p_t = p^N_t / p^T_t\) is the price of the human capital in terms of the price of the consumable physical capital good and \(q_t = 1 / p^T_t\) is the price of money relative to the \(Y^T\) good. We finally obtain that the intertemporal maximization problem of the representative agent can be rewritten as

\[
\max_{[C_t, K_{t+1}, H_{t+1}, M_{t+1}, B_{t+1}]} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1 - \sigma} \tag{18}
\]
subject to (16) and (17). Let now define $\lambda_t$ and $\mu_t$ the Lagrangian multipliers associated, respectively, to the budget constraint and to the CIA constraint. Then the optimal conditions for the representative household with respect to the indicated variables are

\[ C : \quad \beta^{t+1} C_{t+1}^{-\sigma} = \psi \mu_{t+1} + \lambda_{t+1}, \]  
\[ K : \quad 1 + r_{t+1} - \delta T = \frac{\lambda_t}{\lambda_{t+1}}, \]  
\[ H : \quad \frac{p_{t+1}}{p_t} (1 + w_{t+1} - \delta N) = \frac{\lambda_t}{\lambda_{t+1}}, \]  
\[ M : \quad \frac{q_{t+1}}{q_t} (\mu_{t+1} + \lambda_{t+1}) = \lambda_t, \]  
\[ B : \quad \frac{q_{t+1}}{q_t} (1 + i_{t+1}) = \frac{\lambda_t}{\lambda_{t+1}}. \]  

In order equations (19)-(23) to characterize an optimum, one must also take into account the following transversality conditions

\[ \lim_{t \to +\infty} \beta^t C_t^{-\sigma} K_{t+1} = 0, \lim_{t \to +\infty} \beta^t C_t^{-\sigma} p_t H_{t+1} = 0, \lim_{t \to +\infty} \beta^t C_t^{-\sigma} q_t M_{t+1} = 0. \]  

Equations (19)-(23) are no-arbitrage conditions. In particular (22) establishes that the implicit price of money at time $t$, $\lambda_t$, is equal to its value in the following period plus the value of the implicit dividends $\mu_{t+1}$ this asset will pay off. Whenever dividends are positive, money is not seen as a speculative bubble. Observe that $\lambda_{t+1}$ can be interpreted as the marginal indirect utility of real income in period $t$ but, according to (19), it is not equalized by the individual to the marginal utility of consumption, since part of income cannot be transformed into consumption unless it is first used to purchase money balances. Condition (20) and (21) say that no intertemporal transfers of real income are possible to increase total utility. Conditions (20),(21) and (23) gathered together, establish that the investment in the two capital goods and in government bonds are perfectly substitutable and no bubbles do emerge. Condition (23) represents in particular the Fisher equation. Finally, optimal plans for the single household must satisfy the transversality conditions. By exploiting appropriately conditions (20)-(23), we obtain the Euler equation and the intertemporal arbitrage conditions:

\[ \left( \frac{C_{t+1}}{C_t} \right)^{\sigma} = \left( \frac{1 + i_{t+1}}{1 + \psi i_{t+1}} \right) \frac{\beta_{t+1} \psi_{t+1}}{\beta_t}, \]  

and

\[ \frac{q_{t+1}}{q_t} (1 + i_{t+1}) = 1 - \delta T + r_{t+1} = \left( \frac{p_{t+1}}{p_t} \right) \left( 1 - \delta N + w_{t+1} \right). \]  

Condition (25) ensures an optimal consumption smoothing behavior: the return of decreasing one unit of foregoing consumption in period $t$ allows to increase consumption in period $t + 1$ of an amount depending upon current and future nominal interest rates as well as inflation and reflects the fact that a share of consumption requires a previous investment in money balances. Equation (26) specifies the intertemporal
relative-price adjustment necessary to equalize the net returns both on human and physical capital. We now introduce the following assumption, which ensures that the gross rate of return on capital is higher than the profitability of money holding. This requires, in particular, a sufficiently high inflation rate:

**Assumption 3.** The following inequality holds at all dates:

\[
1 - \delta^T + r_{t+1} = \frac{p_{t+1}}{p_t} (1 - \delta^N + w_{t+1}) > \frac{q_{t+1}}{q_t}.
\]  

(27)

Under Assumption 3, the CIA constraint in (17) binds. We will show in the sequel that condition (27) holds at each steady state of the economies studied and thus, by continuity, in a small neighborhood of the formers.

### 3 The economy under constant money growth rate

In this Section, we describe the competitive equilibrium of the economy under the hypothesis that the Central Bank pegs the money growth at a rate constant and equal to \(\hat{\gamma}\). In particular, we prove the existence of a unique balanced growth rate for the economy and appraise its stability properties in regard, notably, to the fiscal and monetary parameter configurations.

#### 3.1 Equilibrium

In order to pursue our objective, we must first provide the definition of competitive equilibrium for the economy and derive the dynamic equations describing it. This will allow us to prove the existence and uniqueness of the balance growth path and, in turn, to carry out a complete stability analysis of the latter. At equilibrium the representative household maximizes her utility, each of the two representative firms maximizes her profit and all markets clear. At this stage of the paper it is useful to observe that in each period \(t\) there are four markets which need to be simultaneously at equilibrium: the consumable physical capital one, the human capital one, the money one and, eventually, the government bonds one. One immediately verifies that equilibrium in the consumable physical capital good market requires

\[
Y_t^T = K_t + (1 - \delta^T)K_t + C_t + G_t
\]

in each period \(t\) meanwhile the human capital good market equilibrium requires

\[
Y_t^N = H_t + (1 - \delta^N)H_t.
\]

In addition, the money market equilibrium implies

\[
q_{t+1}/q_t = C_{t+1}/(\gamma C_t);
\]

one then immediately verifies that, at equilibrium, the real monetary transfers satisfy

\[
\tau_t = \hat{\gamma} M_t/\gamma.
\]

Once the previous three markets are at equilibrium, by Walras law, also the government bonds market does clear. It is now possible to derive the difference equations describing intertemporal equilibrium by simply denoting \(\pi_{t+1} = q_{t+1}/q_t\) the deflation factor between period \(t\) and period \(t + 1\) and by coupling the first order conditions of the firms’ profit maximization problem and of the households’ utility maximization one with the market clearing conditions. The following expressions are thus immediately obtained:

\[
\frac{p_{t+1}}{p_t} = \frac{1 - \delta^T + r(p_{t+1})}{1 - \delta^N + w(p_{t+1})},
\]

\[
\pi_{t+1} = \frac{C_{t+1}}{C_t} \frac{M_t}{M_{t+1}}.
\]

(28)

(29)
\[ Y_T^N = H_{t+1} - (1 - \delta^N)H_t, \]  
\[ Y_I^T = K_{t+1} - (1 - \delta^T)K_t + C_t + G_t, \]  
\[ \pi_{t+1} [1 + i_{t+1}] = 1 - \delta^T + r(p_{t+1}), \quad \frac{M_t}{M_{t+1}} = \frac{1}{\gamma}, \quad \frac{C_{t+1}}{C_t} = \left[ \frac{\beta(1+i_{t+1})(1+\phi_i)\pi_{t+1}}{1+\phi_{t+1}} \right]^{\frac{1}{\delta}}. \]  

Notice that from the first equation in (32) it is possible to obtain a smooth function \( i_{t+1} = i(p_{t+1}, \pi_{t+1}) \) defined by:

\[ i_{t+1} = i(p_{t+1}, \pi_{t+1}) = \frac{1 - \delta^T + r(p_{t+1})}{\pi_{t+1}} - 1 \]  

and thus is possible to express \( i_{t+1} \) as a function of \( p_{t+1} \) and \( \pi_{t+1} \). Once having introduced equations (28)-(32) describing intertemporal equilibrium, it is possible to define the balanced growth path of the economy as a particular path for the real variables involved in the definition of intertemporal equilibrium such that they grow at a constant sustained rate.

**Definition 1 (Balanced Growth Path under constant money growth).** A balanced growth path (BGP hereafter) for the economy is an equilibrium sequence where the variables \( C_t, H_t, K_t, Y_I^T, Y_T^N \) grow at the same constant factor \( g > 1 \). In addition, the shares \( s_t, u_t \), as well as the relative price \( p_t \), the deflation factor \( \pi_t \) and the nominal interest rate \( i_t \) are constant along the BGP.

For reasons of analytical tractability, it is worthwhile to rewrite the equations defining intertemporal equilibrium in the reduced form, i.e. to operate a normalization in terms of a common variable, which in our specific case will be the human capital \( H_t \). This allows us to obtain a normalized dynamic system from which it is possible to draw instead of a BGP, in correspondence to which all the variables increase without bound, a stationary solution in correspondence to which the normalized variable are at rest. Straightforward manipulations yield the following relationships:

\[ \frac{p_{t+1}}{p_t} = \frac{1 - \delta^T + r(p_{t+1})}{1 - \delta^N + r(p_{t+1})}, \]  
\[ \pi_{t+1} = \frac{C_{t+1}}{C_t} \frac{M_t}{M_{t+1}}, \]  
\[ \frac{Y_T^N}{H_t} = \frac{H_{t+1}}{H_t} \frac{C_{t+1}}{C_t} \frac{1 - \delta^N}{1 - \delta^T}, \]  
\[ \frac{Y_I^T}{H_t} = \frac{K_{t+1}}{H_{t+1}} \frac{C_{t+1}}{C_t} - \frac{1}{\delta^T} \frac{K_t}{H_t} + \frac{C_t}{H_t} + \frac{G_t}{H_t}. \]  

Now we can go through the normalization of the variables involved in our dynamic system. Setting \( c_t = \frac{C_t}{H_t}, k_t = \frac{K_t}{H_t}, y_N(p_t, k_t) = \frac{Y_T^N}{H_t} \) and \( y_I(p_t, k_t) = \frac{Y_I^T}{H_t} \), and assuming that public spending is a fraction
\( \theta \in (0, 1) \) of the consumable good, i.e. \( G_t = \theta Y_t \), the reduced form of system defined by (28)-(31) then writes:

\[
\frac{\rho_{t+1}}{\rho_t} - \frac{1 - \delta^T + r(p_{t+1})}{1 - \delta^N + w(p_{t+1})} = 0,
\]

\[
\pi_{t+1} - \frac{1}{\gamma} \left[ \frac{\beta \gamma_1 [1 + i(p_{t+1}, \pi_{t+1})] [1 + \psi(p_{t+1}, \pi_{t+1})]}{1 + \psi(p_{t+1}, \pi_{t+1})} \right]^{\frac{1}{\gamma}} = 0,
\]

\[
\frac{c_t}{c_{t+1}} - \left[ 1 - \delta^N + y^N (p_t, k_t) \right] \left[ \frac{1 + \psi(p_{t+1}, \pi_{t+1})}{\beta \gamma_1 [1 + i(p_{t+1}, \pi_{t+1})] [1 + \psi(p_{t+1}, \pi_{t+1})]} \right]^{\frac{1}{\gamma}} = 0,
\]

\[
\frac{k_{t+1}}{k_t} - \frac{c_t}{c_{t+1}} - \left[ 1 - \delta^T + (1-\theta)y^N(p_t, k_t) \right] \left[ \frac{1 + \psi(p_{t+1}, \pi_{t+1})}{\beta \gamma_1 [1 + i(p_{t+1}, \pi_{t+1})] [1 + \psi(p_{t+1}, \pi_{t+1})]} \right]^{\frac{1}{\gamma}} = 0,
\]

where \( r(p) \) and \( w(p) \) are given in (9). System defined by (38) represents a four dimensional system in the variables lagged once \((p_{t+1}, c_{t+1}, k_{t+1}, \pi_{t+1}, p_t, c_t, k_t, \pi_t)\): the relative price of the human capital, the consumption to human capital ratio, the physical capital to human capital ratio and the nominal deflation factor. Since the unique predetermined variable is the ratio of the stock of the two capital goods, one has that indeterminacy requires the dimension of the stable manifold to be at least equal to two. Were this the case to occur, one would face more than one possibility to locate the initial conditions for the non-predetermined variables. On the other hand, if the dimension of the stable manifold is exactly one (i.e. there is exactly one stable eigenvalue) the system would be perfectly determinate.

### 3.2 Steady state analysis

Once we have introduced intertemporal equilibrium of the reduced system defined by (38), our first task is to ensure the existence and, possibly, the uniqueness of its stationary solution. Once the latter has been found, the corresponding \( BGP \) is immediately derived by simply exploiting the Euler equation of the consumer maximization problem. At the steady state the reduced variable must be constant, i.e. one needs \( p = p_{t+1} = p_t, k = k_{t+1} = k_t, c = c_{t+1} = c_t \) and \( \pi = \pi_t = \pi_{t+1} \). By inspecting (38), a steady state of the reduce dynamics is thus a four-uple \((p, k, c, \pi) \gg (0, 0, 0, 0)\) satisfying the following system of equations:

\[
w(p) - \delta^N = r(p) - \delta^T (A - I)
\]

\[
\pi - \frac{1}{\gamma} \left[ \pi \beta (1 + i(p, \pi)) \right]^{\frac{1}{\gamma}} = 0 \quad (A - II)
\]

\[
1 - \delta^N + y^N (p, k) = \left[ \pi \beta (1 + i(p, \pi)) \right]^{\frac{1}{\gamma}} \quad (A - III)
\]

\[
1 - \delta^T + (1-\theta)y^N(p, k) - \frac{c_t}{k} = \left[ \pi \beta (1 + i(p, \pi)) \right]^{\frac{1}{\gamma}} \quad (A - IV)
\]

Notice that the growth factor \( g \) of the economy is given by (32) evaluated at the steady state. From the first-order conditions (20)-(21), we get the following relationships:

\[
g = \left[ \beta (1 - \delta^T + r(p)) \right]^{\frac{1}{\gamma}} = \left[ \beta (1 - \delta^N + w(p)) \right]^{\frac{1}{\gamma}} = \left[ \pi \beta (1 + i(p, \pi)) \right]^{\frac{1}{\gamma}}.
\]
In order to obtain unbounded growth, one needs \( g(p, \pi) > 1 \), hypothesis which will included into Assumption 4. Observe in addition that the CIA constraint in (17) binds at the steady state if and only if the nominal interest rate \( i \) is positive, condition requiring, in view of (39), the growth factor \( g \) to be lower than \( \gamma/\beta \). In addition, the growth factor \( g \) must be compatible with the transversality conditions, feature requiring a growth factor not too much large, namely \( g < \beta^{\frac{1}{\sigma-1}} \); in such a case the objective function indeed turns out to be infinitely summable and the No-Ponzi game condition for the government is satisfied. In the light of these considerations, defining \( g = \min\{\gamma/\beta, \beta^{1/\sigma-1}\} \), where \( \gamma \) is the endogenous or exogenous money growth factor, we then assume, in the remainder of the paper, the following inequalities to be satisfied.

**Assumption 4** \( 1 < g < \bar{g} \).

Under Assumption 4, system defined by (38) is consistent with intertemporal equilibrium remaining in a small neighborhood of the steady state. We now prove the existence and the uniqueness of the four- uple \( (p, k(p), c(p), \pi(p)) \) defining the BGP. Following Bond et al. (1996), we introduce the following assumption on the technology, ensuring the existence of an equilibrium where investment in both capital goods is profitable:

**Assumption 5** \( \sup_p (r(p) - w(p)) > \delta^T - \delta^N > \inf_p (r(p) - w(p)) \).

Notice that if the production functions satisfy the Inada conditions, Assumption 5 is satisfied. The following Proposition establishes the existence of a unique BGP:

**Proposition 1** Under Assumptions 1-5, there exists a unique solution \( p \) of (A-I) in (39). Then, the four-uple \( (p, k(p), c(p), \pi(p)) \) a unique solution of (39) with \( g(p) > 1 \). Actually, the stationary values of \( k(p), c(p), \pi(p) \) and \( g(p) \) are given by:

\[
k(p) = k_T(p) + \left[ g(p) + \delta^N - \delta^T - g(p) \right] \frac{[k^N(p) - k_T(p)]}{y(p)},\]

\[
c(p) = \left[ (1 - \theta)r(p) + 1 - \delta^T - g(p) \right] k(p) + p(1 - \theta) \left[ w(p) + 1 - \delta^N - g(p) \right],\]

\[
\pi(p) = \frac{\beta(1 - \delta^T + r(p))^{1/\gamma}}{\gamma}\]

and

\[
g(p) = \left[ \beta(1 - \delta^T + r(p)) \right]^{1/\gamma} = \left[ \beta(1 - \delta^N + w(p)) \right]^{1/\gamma} = \left[ \pi(p) \beta (1 + \pi(p)) \right]^{1/\gamma}.\]

**Proof:** See Appendix 6.2.

Proposition 1 ensures the uniqueness of the steady state. Some works close to our own, instead, find multiple steady states. For example, Mino (1997) considers a two-sector model with a consumable
physical capital good and a human capital good. Agents face a CIA constraint not only on consumption expenditures but also on investment in physical capital. More in details, Mino (1997) shows that there can emerge two steady states. These results depend crucially on the fact that the cash-in-advance constraint applies on the whole consumable physical capital good. On the other hand, other contributions, as that of Drugeon (2013), within a two-sector endogenous growth model very close to our own although without CIA constraint, find a unique BGP. Notice that by a direct inspection of (39) and (40), under the monetary regime of constant money growth here considered, money turns out to be neutral in the long-run: as a matter of fact, equations (41)–(44) claim that neither the quantity of money nor the money growth rate do influence the steady state values of the physical variables, although the money growth rate implemented by the Central Bank fixes the deflation factor. On the contrary, as we will see in the section devoted to the analysis of the local stability, the amplitude $\psi$ of the CIA constraint influences the transitional dynamics, and can be dramatically responsible for the emergence of local indeterminacy.

3.3 Local dynamics

After having introduced the definition of intertemporal equilibrium in terms of the reduced variables and having proved the existence and the uniqueness of the stationary solution and hence of the BGP, we now turn to the analysis of the transitional dynamics of the system in the neighborhood of the balanced growth path. Namely, we aim at appraising the stability properties of the unique stationary solution by studying the linearized dynamics of the economy around it. Straightforward although tedious computations show that the linearized dynamics around the unique fixed point of system defined by (38) is generated by the following Jacobian matrix:

$$J = \begin{pmatrix} J_{11}^\gamma & 0 & 0 & 0 \\ J_{21}^\gamma & 1 & J_{23}^\gamma & J_{24}^\gamma \\ J_{31}^\gamma & J_{32}^\gamma & J_{33}^\gamma & 0 \\ J_{41}^\gamma & 0 & 0 & J_{44}^\gamma \end{pmatrix}$$

(45)

It follows that the dynamics of the deviations of the variables involved in the definition of intertemporal equilibrium from their stationary solution is given by the linear system

$$\begin{pmatrix} \frac{dP_{t+1}}{P} \\ \frac{d\overline{c}_{t+1}}{\overline{c}} \\ \frac{dK_{t+1}}{k} \\ \frac{d\pi_{t+1}}{\pi} \end{pmatrix} = J \begin{pmatrix} \frac{dP_t}{P} \\ \frac{d\overline{c}_t}{\overline{c}} \\ \frac{dK_t}{k} \\ \frac{d\pi_t}{\pi} \end{pmatrix}$$

(46)

The terms included into the expression of the Jacobian matrix provided in (45) are the followings:

$$J_{11}^\gamma = \frac{1 - \delta^N + w(p)}{1 - \delta^N + w(p) - p \left( \frac{dr}{dp} - \frac{dw}{dp} \right)},$$

$$J_{21}^\gamma = p \left[ \frac{1}{g} \frac{\partial y^N}{\partial P} + \frac{\psi}{\sigma (1 + \psi i)} \frac{\partial \overline{c}}{\partial P} \right] + \frac{\overline{c}_t}{\sigma (1 + \psi i)} \left[ g \frac{\partial \psi}{\partial \overline{c}} + (1 - \psi \pi) \frac{\overline{c}_t}{\partial \pi} \right] + \frac{1 - \psi}{\sigma (1 + \psi i) (1 + \psi \pi)} \frac{\overline{c}_t}{\partial \pi}.$$
\[ J^{\gamma}_{23} = \frac{k}{g} \frac{\partial y}{\partial k}, \quad J^{\gamma}_{24} = \frac{\psi}{\sigma(1 + \psi i)} \frac{\partial i}{\partial \pi} \left[ 1 + \frac{g}{\pi \gamma} \left( 1 + \frac{1 - \psi}{1 + \psi i} \frac{\partial i}{\partial \pi} \right) \right], \]

\[ J^{\gamma}_{31} = \frac{\psi}{g} \left( 1 - \theta k \frac{\partial y}{\partial p} - \frac{\partial y'}{\partial p} \right), \quad J^{\gamma}_{32} = -c \frac{k}{k}, \quad J^{\gamma}_{33} = 1 + \frac{1}{g} \left( 1 - \theta \frac{\partial y}{\partial k} - k \frac{\partial y'}{\partial k} - y k + c \right), \]

and

\[ J^{\gamma}_{41} = \frac{\psi}{g} \left( \psi + \frac{1 - \psi}{(1 + i)^2} \right), \quad J^{\gamma}_{44} = \frac{\psi \pi}{\gamma} \left( 1 + \frac{1 - \psi}{1 + \psi i} \frac{\partial i}{\partial \pi} \right) \left( 1 + \psi i \right), \]

where the expressions for \( \partial i/\partial \pi \) and \( \partial i/\partial p \) follows directly from equation (33). Two points concerned with the properties of the dynamic system defined in (46) are worthwhile to be emphasized, such points being the consequence of the two-factors, two-goods structure of the production side of the economy. First, the model has a block recursive structure, since the dynamics of the relative price is independent on \( c, k \) and \( \pi \) (Bond et al., 1996). This result comes directly from the factor price equalization property of two-sector models, under which the prices of the factor inputs are determined by prices alone, i.e. they are independent on the relative factor supplies. This allows to simplify the analysis of the transitional dynamics. Second, the signs of \( dr(p)/dp, dw(p)/dp, \partial y/\partial k \) and \( \partial y'/\partial k \) do depend on which sector is the most physical capital intensive (the Stolper-Samuelson effect and the Rybzcynski one). These results are well known from the literature on international trade theory (see e.g. Jones, 1965), and are discussed in Lemma 1. All these informations gathered together can be exploited to derive the value of each term of the Jacobian matrix appearing in (45).

In order to analyze the local stability of system defined by (38), we follow the usual procedure consisting in studying the characteristic polynomial \( P(\lambda) \) of the Jacobian matrix (45) and in analyzing its eigenvalues. From a direct inspection of (45) and of its components above introduced, we get the following Proposition:

**Proposition 2** Under Assumptions 1-5, the characteristic polynomial of the Jacobian matrix (45) is defined by

\[ P(\lambda^\gamma) = (\lambda^\gamma - J^{\gamma}_{11})(\lambda^\gamma - J^{\gamma}_{44}) \left( (\lambda^\gamma)^2 - \lambda^\gamma(1 + J^{\gamma}_{33}) + J^{\gamma}_{33} - J^{\gamma}_{23}J^{\gamma}_{32} \right). \]

Moreover, the characteristic roots are

\[ \lambda^\gamma_1 = J^{\gamma}_{11}, \quad \lambda^\gamma_2 = J^{\gamma}_{44}, \quad \lambda^\gamma_{3\pm} = \frac{1 + J^{\gamma}_{33}}{2} \left[ 1 \pm \sqrt{1 + \frac{(J^{\gamma}_{33})^2 + 4J^{\gamma}_{23}J^{\gamma}_{32} - J^{\gamma}_{23}J^{\gamma}_{32}(1 + J^{\gamma}_{33})}{(1 + J^{\gamma}_{33})^2}} \right]. \] (47)

The following two Propositions establish the stability properties of the dynamic system. It is worthwhile recalling to mind that the latter possesses only one predetermined variable, so indeterminacy occurs if and only if the stable manifold has dimension greater than one. We will find two pictures for the local
dynamics, according to the value taken by the inverse of the intertemporal elasticity of substitution $\sigma$. When the latter is lower than the critical value of 2, indeterminacy arises under both a capital intensive consumable physical capital good and a capital intensive human capital good provided the amplitude of the liquidity constraint is set sufficiently low, meanwhile for an intertemporal elasticity of substitution larger than two the steady state of system defined by (38) is locally indeterminate whatever the relative capital intensity across sector is.

**Proposition 3** Under Assumptions 1-5, let $\sigma < 2$. Then there exists a value $\bar{\psi} = \frac{\sigma}{2\sigma + (2 - \sigma)} \in (0, 1)$ of the share $\psi$ of consumption that must be financed by cash such that the BGP is locally indeterminate when $\psi < \bar{\psi}$ and is locally determinate when $\psi > \bar{\psi}$.

Finally, when $\psi$ goes through $\bar{\psi}$, the steady state undergoes a flip bifurcation.

**Proposition 4** Under Assumptions 1-5, let $\sigma \geq 2$. Then the BGP is locally indeterminate.

*Proof*: See Appendix 6.3.

Proposition 3 shows that when $\sigma$ is lower than two, equilibrium is indeterminate if the amplitude of the CIA constraint $\psi$ is small enough. On the other hand, when $\sigma$ is larger than two, the intertemporal equilibrium is always locally indeterminate, as it is claimed in Proposition 4. Notice that such results are independent on whatever the relative capital intensity is. By contrast to a vast literature on optimal growth which shows that cycles and oscillations require a capital intensive consumption good, in our case the phenomenon of indeterminacy is not linked to the technological features of the economy. Compared with Drugeon (2013) economy, our model differs in respect to the fact that the steady state undergoes a flip bifurcation by changing from determinate to indeterminate and not from overdeterminate to determinate.

By inspecting the definition of $\bar{\psi}$, one immediately verifies that it defines an increasing function in $\sigma$, when $0 < \sigma \leq 2$. In fact, the stationary nominal interest rate is given by

$$i = \frac{\gamma(1 - \delta^T + r(p))^{1 - \frac{1}{\beta}}}{\beta^T} - 1.$$  

It follows that, for a given specification of the technology, which in the light of (A-I) in (39) allows to fix $r(p)$, $i$ is an increasing function in $\sigma$. More in details, when $\sigma$ goes to zero, $i$ goes to zero. Meanwhile for $\sigma$ converging to two, $i$ takes a finite value. As a consequence, $\bar{\psi}$ moves from zero, when $\sigma$ equals zero, to one, when $\sigma$ equals two, and the indeterminacy region does improve with $\sigma$. First, for $\sigma$ lower than two, in the $(\sigma, \bar{\psi})$ plane, it lies below the $\bar{\psi}(\sigma)$ line depicted in Figure 1, and then, for $\sigma$ higher than two, it coincides with the whole parameters space. The shaded area represents the indeterminacy region.

This result is consistent with Bosi and Magris (2003) findings which, within a one sector endogenous growth model, show the same critical value for the intertemporal elasticity of substitution: if the latter is larger than two, indeterminacy is bound to prevail for whatever parameter configuration, otherwise it is required an amplitude for the financial constraint low enough. The finding that indeterminacy is easier to emerge when the amplitude of the liquidity constraint is set lower seems at first sight rather counter-intuitive since usually the scope for indeterminacy decreases as soon as the degree of market imperfection is set lower and lower. However, this is surprising only at first sight, since, following Bosi et al. (2005a),
it is possible to understand the mechanism at work leading to indeterminacy, by following a simple mental experiment. To this end, let us consider the equilibrium money market condition together with the Euler equation. Let us also suppose that the system is at time \( t \) at its BGP equilibrium and let us try to construct an alternative equilibrium path that does not violate the transversality condition. For this purpose, assume that agents collectively revise their expectations in reaction to a given sunspot signal and start believing that the foregoing nominal interest rate will undergo a depreciation. It follows that to reestablish the Euler equation, the current nominal interest rate must be driven up. Yet, the magnitude of the required appreciation will depend crucially upon the share \( \psi \) of the liquidity constraint. If the latter is close to one, a slight increase in the current nominal interest rate would be sufficient, its weight in the right hand side of the Euler equation being large relatively to that of next period nominal interest rate. It follows that the steady state will be unstable in forward dynamics, therefore violating the transversality condition. If, conversely, \( \psi \) is set relatively small, the instantaneous appreciation of the current nominal interest rate must be strong enough to compensate the expected depreciation of the next period nominal interest rate. This will generate a convergent, although oscillatory, forward dynamics, which will move the system back to its stationary solution.

\[ \tilde{\psi}(\sigma) \]

\[ \tilde{\psi}(\sigma) \]

\[ \tilde{\psi}(\sigma) \]

Figure 1: Indeterminacy region.

4 The economy under Taylor rules

In the previous sections we have studied the dynamic features of the economy under the hypothesis that the Central Bank implemented a simple monetary rule consisting in pegging the money growth at a constant rate. We now turn to the analysis of another benchmark monetary policy pursued by the Central Bank aiming at pegging the nominal interest rate on the ground of the estimated deviations of realized inflation from a given target, which boils down to behave according to a specific Taylor rule. Following Leeper (1991), we now assume that the Central Bank follows a Taylor feedback rule of the type:

\[ i_{t+1} = \max \left\{ 0, (1 + \tilde{i}) \left( \frac{\pi_{t+1}}{\pi^*} \right)^{-e_i} - 1 \right\} \]  

(48)

where \( \tilde{i} \) is the implicit target for the nominal interest rate, \( \pi^* \) the implicit target for the deflation factor and \( e_i \neq 1 \) the elasticity of the nominal interest rate with respect to inflation. Following Benhabib et al.
(2001) and Schmitt-Grohé and Uribe (2000), we refer to active interest rate feedback rules the case of rules that respond to increases in inflation with a more than one-for-one increase in the nominal interest rate. As a consequence, a passive interest rate feedback rules is such that the nominal interest rate reacts in a less than one-for-one increase in deflation factor. Since we refer to active or passive interest rate feedback rules in this spirit, the elasticity of the nominal interest rate with respect to the deflation factor falls within one of the above definitions according to its magnitude which can be larger than one (active Taylor rules) or lower than one (passive Taylor rules). However, the distinction between active monetary policy and passive monetary one is sometime different. For instance, Schmitt-Grohé and Uribe (2009) refer to a passive monetary in the case the zero lower bound for the nominal interest rate is reached (with low inflation). This case, in our paper, corresponds to the liquidity trap equilibrium (independently on whether \( \varepsilon_i < 1 \) or \( \varepsilon_i > 1 \)). In the present paper, we retain the most traditional definition and refer to active or passive monetary rules according to the magnitude of the elasticity of the nominal interest with respect to the inflation rate. Conversely, when the economy is at the liquidity trap equilibrium, the interest rate does no more react to increases in inflation but sticks to zero, or close to it. As a matter of fact, a zero lower bound for the nominal interest rate is no longer compatible with the assumption of a binding liquidity constraint, since in this case money is no longer dominated by the other real and financial assets. However, without any loss in generality, we can assume that the lower bound for the nominal interest rate is not exactly zero, but some value arbitrarily close to it although strictly positive. Henceforth the assumption of zero lower bound can be viewed as a limit case of a strictly positive lower bound.

Notice that in the Taylor rules we do not include the output gap since in our model is by construction zero (see, e.g., Woodford, 1993) and, in addition, according to several empirical estimates (see, e.g., Clarida et al., 1998 ), its coefficient falls within a range including very small values for many Central Banks. Since the Central Bank pegs the nominal interest rate, it must supply as much money as the household do demand in correspondence to the chosen interest rate. In analogy with what we have done in respect to the economy characterized by a constant money growth rate, in the sequel we will study the existence and, possibly, the multiplicity of the stationary solutions as well as their stability properties under the hypothesis that the monetary policy is conducted by the Central Bank by implementing the Taylor rule defined in (48).

4.1 Equilibrium

The intertemporal equilibrium of the economy under the hypothesis that the monetary policy is conducted according to the Taylor rule (48) is defined by the same set of equations as in the previous case where it was assumed a constant money growth rate: the unique difference is that now the money supply is endogenous since it must be as large as it is needed to implement the desired nominal interest rate. As a matter of fact, we have now that intertemporal equilibrium is defined by a system of equations which are obtained directly from the first-order conditions of the households’ utility maximization problem, of the firms’ profit maximization one and from the market clearing conditions in the four markets which open in each period, namely the two relative to the consumption and investment goods, the money one and the government bonds one. The intertemporal equilibrium of the economy under the Taylor rule is thus completely characterized by the following equations:

\[
\frac{p_{t+1}}{p_t} = \frac{1 - \delta^T + r(p_{t+1})}{1 - \delta^N + w(p_{t+1})} \tag{49}
\]
\[
\pi_{t+1} = \frac{C_{t+1}}{C_t} \frac{H_t}{H_{t+1}} \tag{50}
\]

\[
Y^N_t = H_{t+1} - (1 - \delta^N)H_t, \tag{51}
\]

\[
Y^T_t = K_{t+1} - (1 - \delta^T)K_t + C_t + G_t, \tag{52}
\]

\[
\pi_{t+1} [1 + i(\pi_{t+1})] = 1 - \delta^T + r(p_{t+1}), \quad \frac{C_{t+1}}{C_t} = \left[ \frac{\beta\pi_{t+1}[1 + i(\pi_{t+1})](1 + \psi_{t+1})}{1 + \psi_{t+1}} \right]^{\frac{1}{\beta}}. \tag{53}
\]

Together with the Taylor rule (48). Notice that the unique difference with respect to the model analyzed previously and corresponding to a constant money growth rate, comes from the fact that in the arbitrage equation (53) does appear the nominal interest rate relative to period \( t + 1 \) as a function of the realized deflation rate relative to the same period, according to the Taylor rule. We can thus now introduce the following definition.

**Definition 2** (Balanced Growth Path under Taylor rules) We define a balanced growth path as an equilibrium where the variables \( C_t, H_t, K_t, Y^N_t, Y^T_t \) grow at the same constant factor \( g > 1 \). The shares \( s_t \) and \( u_t \), as well as the relative price \( p_t \), the deflation factor \( \pi_t \), and the nominal interest rate \( i_t \), are constant along the BGP. Finally, money grows at the gross factor \( g/\pi \), where \( \pi \) is the equilibrium deflation factor.

As it was the case in the economy under a constant rate of money creation, also in the present context, in order to obtain stationary solutions instead of balanced growth paths, it is useful to operate a normalization of the variables involved in (49)-(53). We will follow such a strategy by expressing all the real variables appearing in (49)-(53) in terms of human capital in order to obtain the reduced form of the equilibrium system as follows:

\[
\frac{p_{t+1}}{p_t} = \frac{1 - \delta^T + r(p_{t+1})}{1 - \delta^N + w(p_{t+1})}, \tag{54}
\]

\[
\frac{Y^N_t}{H_t} = \frac{H_{t+1}}{C_{t+1}} \frac{C_t}{H_t} - (1 - \delta^N), \tag{55}
\]

\[
\frac{Y^T_t}{H_t} = \frac{K_{t+1}}{C_{t+1}} \frac{H_t}{C_t} - (1 - \delta^T) \frac{K_t}{H_t} + \frac{C_t}{H_t} + \frac{G_t}{H_t}. \tag{56}
\]

We can now set \( c_t = \frac{C_t}{H_t}, k_t = \frac{K_t}{H_t}, Y^N(p(\pi_t), k_t) = \frac{Y^N_t}{H_t} \) and \( Y^T(p(\pi_t), k_t) = \frac{Y^T_t}{H_t} \), and make use of the arbitrage equation (53) to rely the relative price \( p_t \) to the deflation factor \( \pi_t \). It follows that (49)-(52) can be rewritten in the following notation.
of the steady state. Of course, in correspondence to the liquidity trap text. Under such an Assumption, system defined by (57) is thus consistent with intertemporal equilibrium neighborhood of it. This is ensured by Assumption 4, that we assume to still hold even in the present context. Here again, in analogy with the case corresponding to a constant monetary rule, in order the liquidity constraint to be binding, one needs a positive nominal interest rate at the steady state and thus in a small neighborhood of the steady state. Of course, in correspondence to the liquidity trap

\[ \frac{p_{t+1}}{p_t} = \frac{1-\delta^T + r(p_{t+1})}{1-\delta^T + w(p_{t+1})} = 0, \]

\[ \frac{c_{t+1}}{c_t} - \left[ 1 - \delta^N + y^N(p_t,k_t) \right] \left[ \frac{1+\psi(\pi_{t+1})}{\pi_{t+1} \beta(1+\psi(\pi_t))} \right]^{\frac{1}{\beta}} = 0, \]  

(57)

\[ \frac{k_{t+1}}{k_t} \frac{c_{t+1}}{c_t} - \left[ 1 - \delta^T + \frac{(1-\theta)y^T(p_t,k_t)}{k_t} - \frac{c_t}{k_t} \right] \left[ \frac{1+\psi(\pi_{t+1})}{\pi_{t+1} \beta(1+\psi(\pi_t))} \right]^{\frac{1}{\beta}} = 0. \]

The system defined by (57) is composed by three equations in terms of \( p, k \) and \( c \) lagged once: the relative price of the two goods, the consumption to human capital ratio and the physical capital to human capital ratio. In addition, the arbitrage equation (53) relies the relative price \( \pi \) to the deflation factor \( \pi \). Finally, equation (50) allows to derive the endogenous growth factor of money. System (57) represents a three dimensional system in the variables \((c_{t+1}, k_{t+1}, p_{t+1}, c_t, k_t, p_t)\) since the arbitrage condition \( \pi_{t+1} \left[ 1 + i(\pi_{t+1}) \right] = 1 - \delta^T + r(p_{t+1}) \) allows to derive, for each given \( p \), the corresponding value for the deflation rate \( \pi \), in every period \( t \). Since the unique predetermined variable is the ratio \( k \) of the stock of the two capital goods, indeterminacy again requires the dimension of the stable manifold to be larger than one.

### 4.2 Steady state analysis

As usual when one studies dynamic systems, our first task consists in studying the existence, uniqueness or multiplicity, of the stationary solution of the dynamic system defined by the equations included in (57). Since such equations are defined in terms of levels by mean of the operated normalization, a steady state of system (57) is a constant sequence for the variables involved in (57), i.e. a sequence such that \( p = p_t = p_{t+1}, c = c_{t+1} = c_t \) and \( k = k_t = k_{t+1} \) for all \( t \). To find such a steady state, one needs to check for a three-uple \((p, c, k) > (0, 0, 0)\) solving the following system of equations:

\[ w(p) - \delta^N = r(p) - \delta^T \hspace{1cm} (B - I) \]

\[ 1 - \delta^N + y^N(p, k) = \left[ \pi \beta (1 + i(\pi)) \right]^{\frac{1}{\beta}} \hspace{1cm} (B - II) \]

\[ 1 - \delta^T + \frac{(1-\theta)y^T(p, k)}{k} - \frac{c}{k} = \left[ \pi \beta (1 + i(\pi)) \right]^{\frac{1}{\beta}} \hspace{1cm} (B - III) \]

together with the arbitrage condition \( \pi \left[ 1 + i(\pi) \right] = 1 - \delta^T + r(p) \) evaluated at the steady state. Notice that, in view of the previous expressions, the economy growth factor \( g \) is is obtained by exploiting (25) evaluated at the steady state: from the fist-order conditions (20)-(21) of the households maximization problem, we get in fact the following equalities:

\[ g = \left[ \beta(1 - \delta^T + r(p)) \right]^{\frac{1}{\beta}} = \left[ \beta(1 - \delta^N + w(p)) \right]^{\frac{1}{\beta}} = \left[ \pi \beta (1 + i(\pi)) \right]^{\frac{1}{\beta}}. \]  

(59)

Here again, in analogy with the case corresponding to a constant monetary rule, in order the liquidity trap constraint to be binding, one needs a positive nominal interest rate at the steady state and thus in a small neighborhood of it. This is ensured by Assumption 4, that we assume to still hold even in the present context. Under such an Assumption, system defined by (57) is thus consistent with intertemporal equilibrium remaining in a small neighborhood of the steady state. Of course, in correspondence to the liquidity trap
equilibrium the nominal interest rate is by definition anchored to its zero lower bound. However, as we have already said, in order to avoid a configuration in which money is not more dominated by the other assets, we can assume such a lower bound to be not zero, but arbitrarily close to it. By checking for the solutions of (58), as demonstrated in the Appendix, one find that under passive Taylor rules there is either a Leeper equilibrium or a liquidity trap one, meanwhile were the Central Bank to follow active Taylor rule, one would get two different pictures, one corresponding to the absence of any stationary solution, the other to the emergence at the same time of a Leeper equilibrium besides the liquidity trap one.

**Proposition 5** [Existence and Multiplicity] Suppose Assumptions 1-5 are satisfied. Let $\pi_{\text{max}}$ be the value of the deflation factor such that if $\pi < \pi_{\text{max}}$, then the nominal interest rate is positive, meanwhile, if $\pi \geq \pi_{\text{max}}$, it is at the zero lower bound. Then

i] If $\varepsilon_i < 1$ and $\pi_{\text{max}} < 1 - \delta^T + r(p)$, then the unique steady state is the liquidity trap equilibrium;

ii] If $\varepsilon_i < 1$ and $\pi_{\text{max}} > 1 - \delta^T + r(p)$, then the unique steady state is the Leeper equilibrium;

iii] If $\varepsilon_i > 1$ and $\pi_{\text{max}} < 1 - \delta^T + r(p)$, then there exist two steady states: the Leeper equilibrium and the liquidity trap one;

iv] If $\varepsilon_i > 1$ and $\pi_{\text{max}} > 1 - \delta^T + r(p)$, then no steady state does exist.

In cases i] – iii], the unique solution of system (58) is given by the three-uple $(p, c(p), k(p))$ coupled with the growth factor $g(p) > 1$

\[
k(p) = k^T(p) + \left[\frac{[g(p)+\delta^N-1][k^N(p)-k^T(p)]}{\beta^N(p)}\right], \tag{60}\]

\[
c(p) = \left(1 - \delta^T + (1-\theta)r(p) - g(p)\right)k(p) + p(1-\theta)\left(w(p) + 1 - \delta^N - g(p)\right), \tag{61}\]

\[
g(p) = \left[\beta(1 - \delta^T + r(p))\right]^\frac{1}{\varepsilon_i} = \left[\beta(1 - \delta^N + w(p))\right]^\frac{1}{\varepsilon_i} = \left[\pi^L\beta (1 + i(\pi))\right]^\frac{1}{\varepsilon_i} = \left[\pi^{LT}\beta\right]^\frac{1}{\varepsilon_i}, \tag{62}\]

where $\pi^L < \pi^{LT}$ are the equilibrium values of deflation corresponding to the Leeper equilibrium and the liquidity trap, respectively:

\[
\pi^L = \left[\frac{1 - \delta^T + r(p)}{1 - \delta^T + r(p) + i(\pi)}\right]^{\frac{1}{\varepsilon_i}}, \quad (i = i(\pi^L) > 0), \tag{63}\]

and

\[
\pi^{LT} = 1 - \delta^T + r(p), \quad (i = 0). \tag{64}\]

At the BGP the growth factors of money corresponding to the Leeper equilibrium and to the liquidity
trap one are, respectively:

\[
\left( \frac{M_{t+1}}{M_t} \right)^L = g(p)/\pi^L \tag{65}
\]

and

\[
\left( \frac{M_{t+1}}{M_t} \right)^{LT} = g(p)/\pi^{LT}. \tag{66}
\]

**Proof**: See Appendix 6.4.

In the light of the above Proposition, we thus find that under the implementation of the Taylor rule (48), two stationary equilibria may arise: the liquidity trap one, corresponding to the situation in which the nominal interest rate is at the zero lower bound, and the Leeper equilibrium, characterized by a strictly positive interest rate. Notice that the long-run growth factor is the same for both the two equilibria, since it depends, as it emerges in (62), uniquely upon the rental prices of the two capital goods which, in view the first equation in (48), do not depend upon the monetary policy. We obtain thus that in our economy in which the monetary policy is conducted on the ground of a Taylor rule, money is superneutral in the long run. In addition it is worthwhile noticing that, as in the constant money growth rule case, the amplitude of the CIA constraint \( \psi \) does not more affect the equilibrium values \( p, k(p) \) and \( c(p) \). If the monetary policy does not affect the rate of growth of the economy, nevertheless, the two BGP obtained are characterized by different deflation rates and different money growth factors. Actually, the following inequalities do hold: \( \pi^L < \pi^{LT} \) and \( \left( \frac{M_{t+1}}{M_t} \right)^L > \left( \frac{M_{t+1}}{M_t} \right)^{LT} \). In the next Section we shall analyze the local stability of each of the two stationary equilibria obtained. More in details, we shall prove that, differently from the constant money growth rule, the monetary policy and the amplitude of the CIA constraint are neutral also in the short-run, i.e. do not influence the transitional dynamics.

### 4.3 Local dynamics

In this Section we analyze the transitional dynamics of the dynamic system described in (57) in the neighborhood of the two stationary solutions that may emerge, namely the liquidity trap equilibrium and the Leeper one. In order to carry out such a proposal, we follow the standard procedure consisting in examining the linearized dynamic system around each fixed point of (57). Straightforward although tedious computations show that for any non-negative value of the nominal interest rate, the linearized dynamic system defined by (57) around each stationary solution is generated by the Jacobian matrix

\[
J = \begin{pmatrix}
J_{11}^T & 0 & 0 \\
J_{21}^T & 1 & J_{23}^T \\
J_{31}^T & J_{32}^T & J_{33}^T
\end{pmatrix} \tag{67}
\]

and therefore the dynamics of the deviations from each stationary solution is approximated by the linear dynamic system:
\[
\begin{pmatrix}
\frac{dp_{t+1}}{p} \\
\frac{dc_{t+1}}{p} \\
\frac{dk_{t+1}}{k}
\end{pmatrix} = J \begin{pmatrix}
\frac{dp_t}{p} \\
\frac{dc_t}{p} \\
\frac{dk_t}{k}
\end{pmatrix}
\] (68)

The terms included into the definition of the Jacobian matrix (67) are the following:

\[
J_{11}^T = \frac{1 - \delta^N + w(p)}{1 - \delta^N + w(p) - p \left( \frac{\partial r}{\partial p} - \frac{\partial w}{\partial p} \right)},
J_{21}^T = \frac{p}{g} \left( (1 - \theta) k \frac{\partial y^T}{\partial p} - \frac{\partial y^N}{\partial p} \right),
J_{23}^T = -\frac{c}{kg},
J_{31}^T = \frac{p}{g} \left( (1 - \theta) k \frac{\partial y^T}{\partial p} - \frac{\partial y^N}{\partial p} \right),
J_{32}^T = -\frac{k}{g} \frac{\partial y^N}{\partial k},
J_{33}^T = 1 + \frac{1}{g} \left( (1 - \theta) k \frac{\partial y^T}{\partial k} - k \frac{\partial y^N}{\partial k} - \frac{y^T}{k} + \frac{c}{k} \right).
\]

By looking at (67), one notices that, in perfect analogy with the case in which the Central Bank implements a constant money growth rule, the Jacobian matrix exhibits a block recursive structure, due to the two-factors, two-goods structure of the production side of the economy. However, since the Taylor rule links the nominal interest rate to the deflation rate, the dynamic system loses one dimension (cf. the arbitrage equation (53)). As a consequence, the amplitude of the CIA constraint will not influence the transition towards the steady state. The following proposition introduces the characteristic polynomial as well as the eigenvalues of (68).

**Proposition 6** Under Assumptions 1-5, the characteristic polynomial of the Jacobian matrix (68) is defined by

\[
P(\lambda) = (J_{33}^T - \lambda) \left[ \left( \lambda^T \right)^2 - \lambda^T (1 + J_{22}^T) + J_{22}^T - J_{12}^T J_{21}^T \right].
\]

Moreover, the characteristic roots are given by

\[
\lambda_1^T = J_{33}^T, \quad \lambda_{2+}^T = \frac{1 + J_{33}^T}{2} \left[ 1 \pm \sqrt{1 + \frac{(J_{33}^T)^2 + 4 J_{22}^T J_{12}^T-J_{21}^T J_{22}^T (1+J_{33}^T)}}{(1+J_{33}^T)^2} \right].
\] (69)

Let us observe that the elasticity of the nominal interest with respect to deflation, \( \varepsilon_i \), plays a crucial role in determining the existence of two stationary equilibria. When the Taylor rule is passive, there exists a unique equilibrium and its stability properties are summarized in the following proposition:

**Proposition 7** Suppose that Assumptions 1-5 are satisfied. The unique stationary equilibrium corresponding to \( \varepsilon_i < 1 \) is locally determinate.

*Proof:* See Appendix 6.5.

On the contrary, when the Taylor rule is active, a liquidity trap equilibrium besides the Lepeuer equilibrium may emerge. As it is claimed in the following Proposition, however, both fixed point are locally determinate.
Proposition 8 Suppose that Assumptions 1-5 do hold and that $\pi_{\text{max}} < 1 - \delta_T + r(p)$. Then stationary equilibria arising when $e_i > 1$ are locally determinate.

Proof: See Appendix 6.5.

Propositions 7 and 8 show that no matter the relative capital intensities, the liquidity trap and the Leeper equilibrium are determinate in the sense that the stable manifold has dimension 1. However when both the liquidity trap stationary solution and the Leeper one do exist, for a given value $k_0$ for the initial condition, it is possible to locate the values, $c_0$ and $p_0$ for the initial levels for the intensive consumption and the relative price either on the stable manifold converging to the Leeper steady state or on that ensuring the convergence toward the liquidity trap equilibrium. It follows that, despite each of the stationary solutions is locally determinate, the economy exhibits global indeterminacy, as it is stated in the following Proposition.

Proposition 9 Let $e_i > 1$ and $\pi_{\text{max}} < 1 - \delta_T + r(p)$. Then there are two steady states which exhibit saddle path stability.

We have seen that when the Central Bank implements a Taylor rule, both stationary solutions corresponding, respectively, to the Leeper equilibrium and to the liquidity trap one are locally determinate. Although in order to converge to each stationary solution there exists a unique set for the initial conditions for the non predetermined variables, one faces the possibility to jump since the beginning either on the stable branch converging to the Leeper equilibrium, or on that converging to the liquidity trap one. Actually agents will select the initial level for the non predetermined variables on the ground of their expectations; indeed, both stationary solutions turn out to be consistent with some agents beliefs. We face then a case of global indeterminacy. Such observations call into question the analysis of the saddle path connection between the two stationary solutions and as a consequence the selection devices on the ground of which agents coordinate themselves on the stable branch of one or the other stationary solution. If under the hypothesis of a Central Bank pegging a money growth rate the unique BGP may be locally indeterminate, on the other hand, under the hypothesis of a monetary policy conducted according to a feedback Taylor rule, a global indeterminacy picture does arise: there can be indeed two locally isolated BGP each of them being however locally determinate. Notice that under a feedback Taylor rule, the amplitude of the financial constraint does not affect the stability properties of the BGP, in contrast to the case of a constant money growth rate, under which the amplitude of the financial constraint plays a crucial role in determining the stability of the unique BGP.

5 Concluding remarks

In this paper we have studied an economy populated by a representative agent with an additively separable utility function defined over consumption. He or she in each period, besides consuming, invests in physical capital, human capital, government safe bonds and money balances. There are two representative firms producing according to two different technologies, respectively, the consumption and the physical capital investment good and human capital investment good. The coexistence of two capital goods allows for unbounded growth without relying on external effects in production Romer (1986) or productive public spending Barro (1990). The demand of money is motivated by the presence of a liquidity constraint
imposing agents to finance a given fraction of consumption expenditures out of money balances. Such a fraction can be viewed as the inverse of the velocity of circulation of money, as stated by the Cambridge Balance Approach.

We have focused the attention on two different monetary conducts. According to the first one, the Central Bank pegs a constant money growth rate; according to the second one, the Central Bank follows a feedback Taylor rule and pegs the nominal interest rate in response to the gap between the inflation rate and the inflation target. Within all the frameworks considered, we have first carried out a characterization of the existence, uniqueness and multiplicity of the balanced growth paths. We have found that under a pegging of the money growth rate, there exists a unique balanced growth path. On the other hand, when the Central Bank pegs the nominal interest rate in response to the inflation gap according to a Taylor rule, a liquidity trap equilibrium may emerge besides the Leeper one.

The stability properties of the balanced growth paths depend dramatically upon the choice of the monetary rule adopted. Under a constant money growth rate, the balanced growth path tends to be indeterminate as soon as the amplitude of the liquidity constraint is set lower and lower as well as the intertemporal elasticity of substitution in consumption. This seems at first sight a result rather counter-intuitive, since it is usually thought that the scope for indeterminacy decreases as soon as the degree of market imperfection becomes lower. However, our results seem to confirm the Bosi et al. (2005a) according to which, in an exogenous growth two-sector economy, the indeterminacy region widens as soon as one relaxes the amplitude of the liquidity constraint.

In the regime where the Central Bank follows a Taylor rule, two stationary equilibria may exist provided that the degree of aggressiveness of monetary policy is larger than one: both balanced growth paths are locally stable, even though we find a global indeterminacy configuration.

Our model can be easily extended in several interesting directions. For example, one may wonder how the rate of growth of the economy as well as the stability properties of the balanced growth paths obtained are influenced by assuming a small open economy or by considering a two-country model. The answer would depend, intuitively, upon the choice of the trade structure assumed and upon the hypothesis on the capital market features.

6 Appendix

6.1 Proof of Lemma 1

The Stolper-Samuelson effect \( \frac{d\omega_t}{dp_t} \) and \( \frac{dr_t}{dp_t} \) is obtained by analyzing the factor-price frontier. Then we will also prove that the sign of \( \frac{d\omega_t}{dp_t} \), with \( \omega = p^Tw_t \), is equal to the sign \( \frac{d\omega_t}{dp_t} \).

Under Assumption 1, the production function are homogeneous of degree one, and letting \( F^j_K(K^j_t, H^j_t) \) and \( F^j_H(K^j_t, H^j_t) \) denote, respectively, the derivatives \( \frac{\partial F^j_K(K^j_t, H^j_t)}{\partial K^j_t} \) and \( \frac{\partial F^j_H(K^j_t, H^j_t)}{\partial H^j_t} \), one finds that

\[
H^T_t F^T_H(K^T_t, H^T_t) + K^T_t F^T_K(K^T_t, H^T_t) = Y^T_t, \quad H^N_t F^N_H(K^N_t, H^N_t) + K^N_t F^N_K(K^N_t, H^N_t) = Y^N_t. \tag{70}
\]

Using (9), it follows that (70) is now
Let us now consider the Rybzcynscki effect. From the full-employment conditions obtained by the stocks constraint, we obtain:

\[ K_t^T + K_t^N = K_t, \quad H_t^T + H_t^N = H_t. \]

Letting \( k_t = \frac{K_t}{H_t} \), and by dividing the two last expressions by \( H_t \), we then derive that the factor market clearing equations write

\[
\begin{pmatrix}
\frac{k_t^T}{\gamma_T}
\frac{k_T^N}{\gamma_N}
\frac{h_t^T}{\gamma_T}
\frac{h_t^N}{\gamma_N}
\end{pmatrix}
\begin{pmatrix}
y_t^T
y_t^N
\end{pmatrix}
= \begin{pmatrix} k_t \end{pmatrix}
\tag{75}
\]

Differentiating (75) gives

\[
\begin{pmatrix}
\frac{d k_t^T}{\gamma_T}
\frac{d k_t^N}{\gamma_N}
\frac{d h_t^T}{\gamma_T}
\frac{d h_t^N}{\gamma_N}
\end{pmatrix}
\begin{pmatrix}
y_t^T
y_t^N
\end{pmatrix}
+ \begin{pmatrix}
\frac{k_T^T}{\gamma_T}
\frac{k_T^N}{\gamma_N}
\frac{h_T^T}{\gamma_T}
\frac{h_T^N}{\gamma_N}
\end{pmatrix}
\begin{pmatrix}
dy_t^T
dy_t^N
\end{pmatrix}
= \begin{pmatrix} dk_t \end{pmatrix}
\tag{76}
\]

It follows from the envelope theorem that \( Z_1 \) in (76) is equal to zero. Finally, by arranging (76) in terms of \( dy_t^T /dk_t \) and \( dy_t^N /dk_t \) we derive

\[
\frac{dy_t^N}{dk_t} = \frac{1}{\frac{k_T^N}{\gamma_N}(k_T^N - k_t^N)} d y_t^N = - \frac{k_T^T}{\gamma_T} k_T^N, \quad \frac{dy_t^T}{dk_t} = \frac{1}{\frac{k_T^T}{\gamma_T}(k_T^T - k_t^T)},
\tag{77}
\]
We will now show that \( \text{sign}(d\omega_1/dp_1) = \text{sign}(d\omega_1/dp_1) \). Consider the producers’ first-order conditions (9) expressed in intensive form

\[
f^T(k^T_i(p_i)) - p_i f^N(k^N_i(p_i)) = 0
\]

(78)

Equations (78) can be combined in order to obtain the following equation:

\[
f^T(k^T_i(p_i)) - (k^N_i(p_i) - k^T_i(p_i)) f^T(k^T_i(p_i)) - p_i f^N(k^N_i(p_i)) = 0.
\]

(79)

By totally differentiating equation (79) we obtain:

\[
k^T_i(p_i) = \frac{f^N(k^N_i(p_i))}{f^T(k^T_i(p_i))}, \quad k^N_i(p_i) = \frac{f^T(k^T_i(p_i))}{p_i f^N(k^N_i(p_i))}.
\]

(80)

Consider now the definitions of factor rental rates: \( r(p_i) = f^T(k^T_i(p_i)) \) and \( w(p_i) = f^N(k^N_i(p_i)) - k^N_i(p_i) f^N(k^N_i(p_i)) \). Totally differentiating them we obtain:

\[
\begin{align*}
r'(p_i) &= \frac{dr(p_i)}{dp_i} = f^{TT}(k^T_i(p_i))k^T_i(p_i), & w'(p_i) = \frac{dw(p_i)}{dp_i} = -k^N_i(p_i)f^{NN}(k^N_i(p_i))k^N_i(p_i),
\end{align*}
\]

(81)

where \( k^T_i(p_i) \) and \( k^N_i(p_i) \) are given by equations (80). This proves that \( \text{sign}(d\omega_1/dp) = \text{sign}(d\omega_1/dp) \).

### 6.2 Proof of Proposition 1

System (39) is four dimensional in \( p, k, c, \pi \). In the following, we show that there exists a unique and positive four-uple \((p, k, c, \pi)\) satisfying (39).\(^3\) Let us first consider (A-I) in (39) and define \( Z(p) = r(p) - w(p) \). By differentiating it and using equations (81), one can show that it is a monotone function of \( p \). Namely, \( Z(p) \) is decreasing (resp. increasing) if the consumable good is physical capital intensive (resp. human capital intensive). This proves the uniqueness of the solution. Assumption 5 guarantees the existence.\(^4\)

By replacing \( p \) by its particular positive value in (A-II), (A-III) and (A-IV), we will next show that there exists a unique and positive \( \pi(p), k(p) \) and \( c(p) \) satisfying, respectively, (A-II), (A-III) and (A-IV) in (39). By using (40) in (A-II), we obtain the unique positive value of \( \pi(p) \) corresponding to equation (43) in the text.

We show now that there exists a unique and positive \( k(p) \) consistent with balanced growth. Under diversification it holds that \( k(p) \in \{\min[k^N(p), k^T(p)], \max[k^N(p), k^T(p)]\} \). \( k(p) \) is feasible if and only if \( 0 < g(p) + \delta^N - 1 < y^N(p) \). Since \( y^N(p) > w(p) \), condition \( w(p) + 1 - \delta^N > g(p) \) is sufficient for the existence of a unique and positive solution \( k(p) \). By using equation (40), it is possible to rewrite this inequality as \( \beta < 1/g^{1-\sigma} \). It is easy to verify that this is satisfied if \( \sigma \leq 1 \). Assumption 2 assures that this

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\(^3\)The rest of the proof is based upon arguments similar to the ones introduced in Bond et al. (1996) in Lemma 1 and Proposition 1.

\(^4\)One can also prove this by considering that under Assumption 1, Lemma 1 and a physical capital intensive consumable capital good (resp. human capital intensive consumable capital good), the left-hand side of (A-I) is a monotone increasing (resp. decreasing) function in \( p \) from \( -\delta^N \) to \( +\infty \) (resp. \( +\infty \) to \( -\delta^N \)), while the right-hand side of (A-I) is a monotone decreasing (resp. increasing) function moving from \( +\infty \) to \( -\delta^T \) (resp. \( -\delta^T \) to \( +\infty \)). By Assumption 5, it follows that there exists a unique \( p \) solution of (A-I) of (39).
inequality is satisfied for all values of $\sigma$. Hence, there will exists a unique and positive $k(p)$ defined by (41) consistent with balanced growth.

Finally, we prove that there exists a unique and positive $c(p)$ consistent with the BGP. Constant returns to scale in both sectors imply that $y^T + py^N = pw(p) + r(p)k(p)$. Using this expression together with (A-IV) in (39), we obtain (42), which is positive in view of the above arguments. It follows that $c(p)$ is the unique positive solution of (A-IV) in (39).

By replacing the steady state value $p$ in (40), we then obtain a unique growth factor $g$, which is larger than 1 under Assumption 4. The BGP solutions satisfy the transversality conditions defined in (24). This is guaranteed by Assumption 2 and Assumption 4.

6.3 Proof of Propositions 3 and 4

In order to study the local stability of the stationary solution, we must analyze the four characteristic roots defined in Proposition 2 by (47). Let us first consider $\lambda_1^T = J_{11}^T$ which is given by:

$$\lambda_1^T = \frac{1 - \delta^N + w(p)}{1 - \delta^N + w(p) - p \left( \frac{dr}{dp} - \frac{dw}{dp} \right)}.$$  \hspace{1cm} (82)

From Lemma 1, we know that if $k^T > k^N$ then $d\omega/dp > 0$ and $dr/dp < 0$, meanwhile if $k^T < k^N$ then $d\omega/dp < 0$ and $dr/dp > 0$. When $k^T > k^N$, the denominator of (82) is higher than its numerator in absolute value, implying that $|\lambda_1^T| < 1$. When $k^T < k^N$, the denominator of (82) is lower than its numerator in absolute value, implying that $|\lambda_1^T| > 1$. Let us now consider $\lambda_2^T = J_{14}^T$. From (45) and equation (33) we can write

$$\lambda_2^T = -\frac{\psi(1 + i)}{(1 + \psi i) \left[ 1 - \sigma - \frac{1 - \psi}{1 + \psi i} \right]}.$$  \hspace{1cm} (83)

To analyze $\lambda_2^T$, we observe that this characteristic root is parametrized by $\psi \in (0, 1)$ and then we can study its behavior according to the admissible range for $\psi$. Is it immediate to verify the following three properties

$$\lim_{\psi \to 0} \lambda_2^T = 0, \quad \lim_{\psi \to 1} \lambda_2^T = \frac{1}{1 - \sigma}, \quad \frac{\partial \lambda_2^T}{\partial \psi} = -\frac{(1 + i)\sigma}{(1 + \psi i)^2 \left[ 1 - \sigma - \frac{1 - \psi}{1 + \psi i} \right]^2}.$$  \hspace{1cm} (85)

It follows that the characteristic root is a monotone decreasing function of $\psi$ moving from 0 to 1/(1 $- \sigma$). The only remaining task is to verify if $\lambda_2^T$ crosses 1 or and -1. By setting $\lambda_2^T = 1$, it is immediate to verify that such a result is not possible. Setting $\lambda_2^T = -1$ gives:

$$\tilde{\psi} = \frac{\sigma}{2 + i(2 - \sigma)}.$$  \hspace{1cm} (86)

This value $\tilde{\psi}$ must fall within the admissible values for $\psi \in (0, 1)$. It is immediate to verify that when $\sigma < 2$ we get $\tilde{\psi} \in (0, 1)$. Then, when $\psi < \tilde{\psi}$ we get $|\lambda_2^T| < 1$, meanwhile when $\psi > \tilde{\psi}$ we get $|\lambda_2^T| > 1$. Moreover, when $\sigma \geq 2$, then $|\lambda_2^T| < 1$ for all $\psi$, since $\lambda_2^T$ is a monotone decreasing function of $\psi$ going from 0 to 1/(1 $- \sigma$).

Finally, let consider the two characteristic roots $\lambda_3^T$. In order to analyze them, we study the Trace
and the Determinant of the 2X2 subsystem corresponding to the 2X2 sub-jacobian matrix of system (45) defined by

\[ J^y_2 = \begin{pmatrix} 1 & J^y_{32} \\ J^y_{32} & J^y_{33} \end{pmatrix}. \]

The corresponding characteristic polynomial is given by:

\[(\lambda^2 - (1 + J^y_{33})\lambda + (J^y_{33} - J^y_{23}J^y_{32})) = 0. \quad (84)\]

where \( T(J^y_2) = 1 + J^y_{33} \) and \( D(J^y_2) = J^y_{33} - J^y_{23}J^y_{32} \) are the trace and the determinant of the matrix \( J^y_2 \), respectively. In the case \( k^T > k^N \), the trace is greater than one and the determinant is less than one. It follows that \( D(J^y_2) > T(J^y_2) - 1 \) and \( D(J^y_2) < -T(J^y_2) - 1 \). These conditions implies that the real part of the eigenvalues \( \lambda^T_3 \geq 1 \) is greater than 1. In the case \( k^T < k^N \), the trace and the determinant are greater than one. It follows that \( D(J^y_2) > T(J^y_2) - 1 \) and \( D(J^y_2) < -T(J^y_2) - 1 \). These conditions implies that one of the real part of the eigenvalues \( \lambda^T_3 \geq 1 \) is greater than 1 and the other is less than 1.

### 6.4 Proof of Proposition 5

As we have shown in Appendix 6.2, equation (49) defines a unique equilibrium value of the relative price \( p \). The value of \( k(p) \) and \( c(p) \) are determined as in the constant money growth case. This value determines the value of the deflation factor \( \pi \) from the arbitrage condition (53). The analysis of this equation deserves a deep inspection. The Taylor rule (48) puts an upper bound on \( \pi \), that we will refer as to \( \pi_{\text{max}} = (1 + i^*)^{\frac{1}{\delta^T}} \pi^* \), such that if \( \pi < \pi_{\text{max}} \), then the nominal interest rate is positive, \( i = i(\pi) \), while if \( \pi \geq \pi_{\text{max}} \), then it is at the zero lower bound. Consider the arbitrage equation (53) at the steady state:

\[ G(\pi) \equiv \pi \{1 + i(\pi)\} = 1 - \delta^T + r(p). \]

The right hand side is known once one has computed \( p \) from equation (49). If \( \pi \geq \pi_{\text{max}} \), then \( G(\pi) = \pi \). In the opposite case, \( G(\pi) = \pi^{1 - \delta^T} (1 + i^*)^{\frac{1}{\delta^T}} \pi^* \). This function is increasing in \( \pi \) for \( \varepsilon_i < 1 \) and decreasing for \( \varepsilon_i > 1 \); in this last case one has that \( \lim_{\pi \to 0} = +\infty \). It follows that the function \( G(\pi) \) has the behavior depicted in Figure 2. Four cases may now arise. Let us suppose first that \( \varepsilon_i < 1 \). In this case there exists a unique steady value of \( \pi \): it will be that of the liquidity trap when \( \pi_{\text{max}} < 1 - \delta^T + r(p) \), and of the Leeper equilibrium when \( \pi_{\text{max}} > 1 - \delta^T + r(p) \). When \( \varepsilon_i > 1 \), multiplicity of the steady state values of \( \pi \) may conversely arise. In particular, when \( \pi_{\text{max}} < 1 - \delta^T + r(p) \), the function \( G(\pi) \) intersects twice the line \( 1 - \delta^T + r(p) \) and two equilibria arise: the Leeper equilibrium and the liquidity trap one. In the opposite case, no stationary solution does exist.

In the case where at least an equilibrium value of \( \pi \) exists, it is possible to show that the equilibrium value of \( c \) and \( k \) exist, by using the same arguments as in Appendix 6.2. These values are the same in the two equilibria since equation (49) determines a unique value of the relative price \( p \). It follows also that the growth rate \( g(p^*) \) is the same in the two regimes, as emerges from relation (59). In fact, the arbitrage equation (53) establishes the relation \( \pi^T(1 + i(\pi^T)) = 1 - \delta^T + r(p) = \pi^LT \).
Finally, equation (53) determines the growth rate of money, which is different in the Leeper equilibrium and in the liquidity trap, as it is shown by equations (65) and (66).

\[ \pi G(\pi) = 1 - \delta^T + r(p) \]

\[ \pi \max \]

\[ \begin{array}{c}
\text{(a) } \varepsilon_i < 1. \\
\text{(b) } \varepsilon_i < 1.
\end{array} \]

\[ \begin{array}{c}
\text{(c) } \varepsilon_i > 1. \\
\text{(d) } \varepsilon_i > 1.
\end{array} \]

Figure 2: Steady states.

6.5 Proof of Propositions 7 and 8

From the previous arguments included in Appendix 6.3, we know that \( k^T > k^N \) implies that \( |\lambda_1^T| \) is smaller than 1 and the real part of the eigenvalues \( \lambda_{2 \pm}^T \) is greater than 1, meanwhile for \( k^T < k^N \), \( |\lambda_1^T| \) is greater than 1 and the real part of one the eigenvalues \( \lambda_{2 \pm}^T \) is greater than 1 and the other is less than 1.

References


